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(5)

APPLICATION OF A DEFINITE INTEGRAL INVOLVING BESSEL'S FUNCTIONS TO THE SELF-INDUCTANCE OF SOLENOIDS.

BY PROFESSOR ARTHUR GORDON WEBSTER.

(Read before the American Mathematical Society, December 29, 1905).

THE construction in the Physical Laboratory of Clark University of two primary standards of self-inductance by Dr. J. G. Coffin and his development of satisfactory formulas for the calculation of their values[*] led the writer to attempt to find an expression of a different sort for this quantity, namely the self-inductance of a current sheet in the form of a circular cylinder, practically represented by a wire wound in a close solenoid of a single layer.[†] All the methods that have been previously used have involved elliptic integrals or developments in spherical harmonics. A quite different mode of approaching the problem was suggested by a paper by Nagaoka,[‡] in which use is made of the following integral given by H. Weber[§] for the value of the potential of a circular disk of radius a and of unit surface density at a point at a distance z from its plane and r from the axis of symmetry:

$$V = 2\pi a \int_0^\infty \frac{e^{-\lambda z}}{\lambda} J_0(\lambda r) J_1(\lambda a) d\lambda \qquad (z > 0). \tag{1}$$

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[*] J. G. Coffin, *Bulletin* of the Bureau of Standards, vol. 2, No. 1 (June, 1906).

[†] Since this paper was read several papers by E. B. Rosa and L. Cohen on
tance of such solenoids have appeared in the *Bulletin* of the
ds.

the potential and lines of force of a circular current,"
. 19, 1903.

"Ueber die Bessel'schen Funktionen und ihre Anwendungen
heorie der elektrischen Ströme." *Crelle's Journal*, vol. 75, p. 75.

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The magnetic field due to a current I in a close sol m turns per unit of length parallel to the axis, which take as the z-axis, is $4\pi mI$, if the solenoid is of infinite l If not, we can find the effect of the ends by placing at the tive and negative ends disks with surface density $\pm mI$ res tively. We have then for the magnetic potential at points side the solenoid

$$(2) \qquad \Omega = mI(V_1 - V_2 - 4\pi z),$$

and for the field

$$(3) \qquad H = -\frac{\partial \Omega}{\partial z} = mI\left(4\pi - \frac{\partial V_1}{\partial z} + \frac{\partial V_2}{\partial z}\right),$$

in which V_2 denotes the value obtained by putting in (1) for the distance z_2 of the point from the negative end of the cylinder and V_1 the value obtained by putting the distance from th positive end $z_1 = l - z_2$, where l is the length of the solenoid

Differentiating under the integral sign, since

$$\frac{\partial}{\partial z_2} = -\frac{\partial}{\partial z_1} = \frac{\partial}{\partial z},$$

we find the flux through a cross-section of the cylinder b multiplying by $2\pi r dr$ and integrating from 0 to a,

$$(4) \qquad \begin{aligned} 2\pi\int_0^a Hrdr = mI\Big\{ & 4\pi^2a^2 - 4\pi^2 a\int_0^a rdr\int_0^\infty e^{-\lambda z_1}J_0(\lambda r)J_1(\lambda a)d \\ & - 4\pi^2 a\int_0^a rdr\int_0^\infty e^{-\lambda z_2}J_0(\lambda r)J_1(\lambda a)d\lambda\Big\} \end{aligned}$$

Now we have

$$\int_0^a rJ_0(\lambda r)dr = \frac{1}{\lambda^2}\int_0^{\lambda a}\xi J_0(\xi)d\xi = \frac{1}{\lambda^2}\left[\xi J_1(\xi)\right]_0^{\lambda a} = \frac{a}{\lambda}J_1(\lambda a),$$

so that, changing the order of integration, we obtain for th flux through the circular section

$$(5) \qquad 4\pi^2a^2mI\left\{1 - \int_0^\infty \frac{e^{-\lambda z_1}}{\lambda}J_1^2(\lambda a)d\lambda - \int_0^\infty \frac{e^{-\lambda z_2}}{\lambda}J_1^2(\lambda a)d\lambda\right\}$$

Since the length dz carries the current $mIdz$, m by the flux just found and integrating for the w we obtain for the self-inductance L

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$$(6)\qquad LI^2 = 4\pi^2a^2m^2I^2\left\{l - 2\int_0^l dz\int_0^\infty \frac{e^{-\lambda z}}{\lambda}J_1^2(\lambda a)d\lambda\right\}.$$

We may now interchange the order of integration, performing the integration according to z,

$$(7)\qquad L = 4\pi^2a^2m^2\left\{l + 2\int_0^\infty (e^{-\lambda l} - 1)\frac{J_1^2(\lambda a)}{\lambda^2}\,d\lambda\right\}.$$

We may arrive more quickly at the result (7) by starting from the fundamental definition of the mutual inductance $M_{xx'}$ of two circular lines situated at distances x and x' from one end of the cylinder, both of which are made to occupy all positions possible, when

$$(8)\qquad L = \int_0^l\int_0^l M_{xx'}dx\,dx',$$

$$(9)\qquad \begin{aligned} M_{xx'} &= \int\int\frac{ds\,ds'\cos\epsilon}{r} \\ &= 2\pi aa'\int_0^{2\pi}\frac{\cos\theta\,d\theta}{\sqrt{(x-x')^2 + a^2 + a'^2 - 2aa'\cos\theta}}. \end{aligned}$$

In order to transform the elliptic integral (9) we make use of the integral of Lipschitz (Gray and Mathews, page 72)

$$(10)\qquad \int_0^\infty e^{-\lambda z}J_0(\lambda\omega)d\lambda = \frac{1}{\sqrt{z^2+\omega^2}},$$

and the addition formula of Neumann (Gray and Mathews, page 27, 69′)

$$(11)\qquad J_0(\lambda\omega) = J_0(\lambda a)J_0(\lambda a') + 2\sum_{n=1}^{n=\infty}J_n(\lambda a)J_n(\lambda a')\cos n\theta,$$

where

$$\omega^2 = a^2 + a'^2 - 2aa'\cos\theta,$$

which being multiplied by $\cos\theta$ and integrated from 0 to 2π gives

$$(12)\qquad \int_0^{2\pi}J_0(\lambda\omega)d\theta = 2\pi J_1(\lambda a)J_1(\lambda a').$$

Treating (10) in the same manner, and using this result, we convert (9) into

$$(13) \qquad M_{xx'} = 4\pi^2 aa' \int_0^\infty e^{-\lambda|x-x'|} J_1(\lambda a) J_1(\lambda a') d\lambda.$$

We may now perform the integration with regard to x' and then x, taking care to notice that the cofactor of $-\lambda$ in the exponential must be positive, so that we put $x - x' = z$ when $x > x'$ and $x - x' = -z$ when $x < x'$, giving

$$\int_0^l M_{xx'} dx'$$
$$= 4\pi^2 aa' \int_0^\infty J_1(\lambda a) J_1(\lambda a') d\lambda \left[\int_0^x e^{-\lambda z} dz + \int_0^{l-x} e^{-\lambda z} dz \right]$$

$$[\cdots] = \frac{1}{\lambda}(2 - e^{-\lambda x} - e^{-\lambda(l-x)}), \quad \int_0^l [\cdots]\, dx = \frac{2}{\lambda}\left(l + \frac{e^{-l\lambda} - 1}{\lambda}\right).$$

$$(14) \qquad \int_0^l \int_0^l M_{xx'} dx dx' =$$
$$8\pi^2 aa' \int_0^\infty \left(\frac{l}{\lambda} + \frac{e^{-\lambda l} - 1}{\lambda^2} \right) J_1(\lambda a) J_1(\lambda a') d\lambda.$$

This formula serves for the mutual inductance of two solenoids of the same length but of different diameters, but for our purpose $a = a'$, so that we again obtain (7), since

$$\int_0^\infty \frac{J_1^2(t)}{t} dt = \frac{1}{2}.$$

Let us now introduce the variable $t = \lambda a$ and the constant $z = l/a$ for the ratio of the length to the radius of the solenoid, so that

$$(15) \quad L = 8\pi^2 a^3 m^2 \left[\frac{z}{2} - \int_0^\infty \frac{J_1^2(t)}{t^2} dt + \int_0^\infty e^{-zt} \frac{J_1^2(t)}{t^2} dt \right].$$

Of these two definite integrals, the first is a constant, whose value we find by the formula given by Nielsen, Handbuch der Theorie der Cylinderfunktionen, page 194, (15) to be $4/3\pi$, while the second is a function of z which we will develop into a power series.

$$(16) \qquad Z = \int_0^\infty e^{-zt} \frac{J_1^2(t)}{t^2} dt.$$

Now we have (Nielsen, page 20, (4)),

$$(17)\qquad J_1^2(t) = \sum_0^\infty \frac{(-1)^s(2s+2)!}{s!\,(s+1)!\,(s+1)!\,(s+2)!}\left(\frac{t}{2}\right)^{2s+2},$$

so that

$$(18)\qquad Z = \sum_0^\infty \frac{(-1)^s(2s+2)!}{2^{2s+2}s!\,(s+1)!\,(s+1)!\,(s+2)!}\int_0^\infty e^{-zt}t^{2s}\,dt.$$

The definite integral is now a gamma function, whose value is easily seen to be

$$(2s)!/z^{2s+1}.$$

Thus we obtain

$$(19)\qquad Z = \sum_0^\infty \frac{(-1)^s(2s)!\,(2s+2)!}{2^{2s+2}s!\,(s+1)!\,(s+1)!\,(s+2)!}\,\frac{1}{z^{2s+1}},$$

a series which is convergent when $z > 2$. We have accordingly the final result

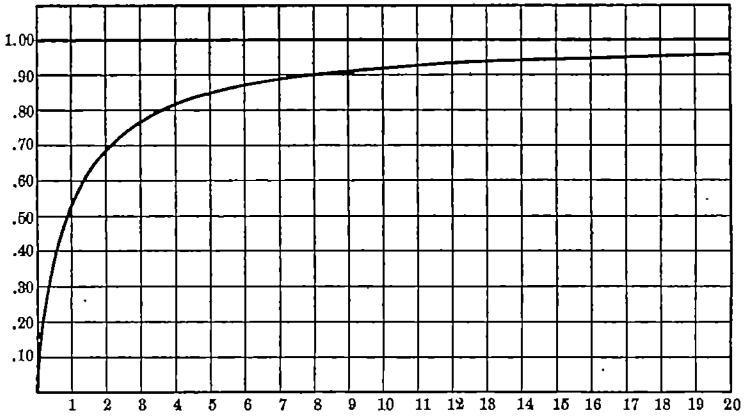
$$(20)\qquad \begin{aligned} L &= 4\pi^2a^3m^2\left\{z - \frac{8}{3\pi}\right. \\ &\qquad\left. + \sum_0^\infty \frac{(-1)^s(2s)!\,(s+2)!}{2^{2s+1}s!\,(s+1)!\,(s+1)!\,(s+2)!}\,\frac{1}{z^{2s+1}}\right\} \\ &= 4\pi^2a^3m^2\left\{z - \frac{8}{3\pi} + \frac{1}{z} - \frac{1}{2z^3} + \frac{5}{8z^5} - \frac{35}{32z^7} + \frac{147}{64z^9}\right. \\ &\qquad\left. - \frac{693}{128z^{11}} + \frac{14{,}157}{1{,}024z^{13}} - \cdots\right\}, \end{aligned}$$

giving the inductance directly as a power series in terms of the ratio of length to radius, when the length of the cylinder is greater than its diameter. The series does not converge rapidly until the length is several times the diameter, but Coffin's formula

$$(21)\qquad \begin{aligned} L = 4\pi an^2\Big\{&(N - \tfrac{1}{2}) + \frac{z^2}{32}(N + \tfrac{1}{4}) - \frac{z^4}{1{,}024}(N - \tfrac{2}{3}) \\ &+ \frac{10z^6}{131{,}072}(N - \tfrac{109}{120}) - \frac{35z^8}{4{,}194{,}304}(N - \tfrac{431}{420}) \cdots\Big\}, \end{aligned}$$

$$\left(N = \log_e \frac{8a}{l},\; n = lm\right),$$

converges better the shorter the cylinder, so that by one of these formulas the calculation may always be made. Coffin has also given a formula by elliptic integrals, which is convenient when tables are at hand. Mr. Gordon Fulcher has calculated by the three methods values for some twenty ratios, from which he has constructed the annexed graph of $L/4\pi^2a^3m^2z^2$, that is the factor of correction for the ends of the solenoid. When the length is ten times the diameter, five terms of (20) give nine figures of the result.



CLARK UNIVERSITY,
July, 1907.

ON THE APSIDAL ANGLE IN CENTRAL ORBITS.

BY DR. F. L. GRIFFIN.

(Read in part before the American Mathematical Society, April 27, 1907.)

THERE are two well known laws of central force all of whose trajectories have the same apsidal angle, whatever be the apsidal values of the radius vector, viz., that of Newton and the law that the force varies directly as the distance. For both of these laws the orbits are all conic sections, the apsidal angle being π in the former case, and $\frac{1}{2}\pi$ in the latter. Generally, however, the apsidal angle depends upon the apsidal values of the radius vector.

In this paper are considered only those laws of central force for which the force is a function of the distance, having a finite ivative except at a finite number of distances in any inter- .. For all such laws, and for all orbits lying wholly within y interval throughout which this derivative is finite, four eorems will be proved. If, for a given law, the derivative finite for all values of the distance, the theorems hold for all e trajectories. The criteria obtained find immediately two plications: (1) To orbits in the equatorial planes of attracting heroids—incidentally explaining a well known phenomenon the motion of the fifth satellite of Jupiter; and (2) to the ajectories for various simple laws of force.

Let u denote the reciprocal of the radius vector, and $u^2P(u)$, ıe force; and let $P'(u) = d[P(u)]/du$. Evidently, the deriva- ve of the force with respect to the distance is finite if, and nly if, $P'(u)$ is finite.

THEOREM I. *In order that the apsidal angle be greater than* r (equal to π, or less than π) *in every trajectory for a given law,* t *is necessary and sufficient that* $P'(u)$ *be* (respectively) *every- vhere positive* (zero, or negative).

The proof is got by considering the definite integral which gives the apsidal angle. If h be the constant of areas and θ the longitude, the differential equation of any orbit under the given law is

$$h^2u^2(u + d^2u/d\theta^2) = u^2P(u) \equiv \tfrac{1}{2}u^2F'(u).$$

[$P(u)$, since it has a derivative, is continuous and therefore integrable.] After integration, one has

$$h^2[u^2 + (du/d\theta)^2] = c + F(u),$$

c being a constant. Or, if α and β [$\beta > \alpha$] be the apsidal values of u,

$$(1)\qquad \begin{aligned} \left(\frac{du}{d\theta}\right)^2 &= (\beta^2 - u^2) + \frac{1}{h^2}[F(u) - F(\beta)] \\ &= (\alpha^2 - u^2) + \frac{1}{h^2}[F(u) - F(\alpha)]. \end{aligned}$$

Hence the apsidal angle is given by

$$(2)\qquad \Theta = \int_\alpha^\beta \frac{du}{\sqrt{\beta^2 - u^2 + \frac{1}{h^2}[F(u) - F(\beta)]}}.$$

Sufficiency of the condition on $P'(u)$. Compare Θ with the apsidal angle in the case of Newton's law, where $P(u) =$ a constant. For the latter law the second member of (1) $(\beta - u)[\beta + u - 2N/h^2]$. Hence, if N be selected,

$$N = \tfrac{1}{2}h^2(\alpha + \beta),$$

the Keplerian ellipse will have the same constant of areas an the same apsidal values of u as the original orbit. Therefo

$$(3) \qquad \pi = \int_\alpha^\beta \frac{du}{\sqrt{\beta^2 - u^2 + \frac{2N}{h^2}(u - \beta)}},$$

$$(4) \qquad \Theta - \pi = \int_\alpha^\beta \left\{ \frac{1}{\sqrt{\beta^2 - u^2 + \frac{1}{h^2}[F(u) - F(\beta)]}} - \frac{1}{\sqrt{\beta^2 - u^2 + \frac{2N}{h^2}(u - \beta)}} \right\} du.$$

Consider now the function

$$h^2\left\{\beta^2 - u^2 + \frac{2N}{h^2}(u - \beta)\right\} - h^2\left\{\beta^2 - u^2 + \frac{1}{h^2}[F(u) - F(\beta)]\right\},$$

or

$$(5) \qquad \Phi(u) \equiv 2N(u - \beta) - F(u) + F(\beta).$$

Since in both orbits $du/d\theta$ vanishes at $u = \alpha$, and at $u = \beta$, it follows that

$$(6) \qquad \Phi(\alpha) = \Phi(\beta) = 0.$$

Also,

$$\Phi''(u) = -F''(u) = -2P'(u).$$

Suppose now that $P'(u) > 0$ for $\alpha \leqq u \leqq \beta$. Then $\Phi''(u) < 0$, and consequently $\Phi'(u)$ decreases throughout the interval (α, β). Now since $\Phi'(u)$ has a derivative it is continuous, and therefore vanishes at most once for $\alpha \leqq u \leqq \beta$. But, from (6), $\Phi'(u)$ must vanish once, by Rolle's theorem. Since $\Phi'(u)$ decreases from $u = \alpha$ to $u = \beta$ and vanishes between α and

β, $\Phi'(\alpha) > 0$ and $\Phi'(\beta) < 0$. Consequently $\Phi(u) > 0$ for $\alpha < u < \beta$.

This shows that the integrand (4) is everywhere positive, whence $\Theta - \pi > 0$ or $\Theta > \pi$.

Similarly, suppose $P'(u)$ everywhere negative; then $\Phi(u) < 0$, the integrand is everywhere negative, and $\Theta < \pi$. Likewise for $P'(u) = 0$, the integrand is zero throughout, and $\Theta = \pi$. Thus, the sufficiency of the condition is established.

Necessity of the condition on $P'(u)$. Suppose $P'(u)$ changes sign within the interval (α, β). Then within some interval (α_1, β_1), $P'(u) > 0$; and within some interval (α_2, β_2), $P'(u) < 0$. Hence, by the sufficiency proof, all orbits lying wholly within (α_1, β_1) have their apsidal angles greater than π; and all those lying within (α_2, β_2) have their apsidal angles less than π. But this does not fulfill the requirement that Θ be greater than π (equal to π, or less than π) for *all* orbits lying within (α, β). Hence the condition is necessary.

THEOREM II. *If $P'(u)$ is positive, and increases with u or is constant, then, in those families of orbits one of whose apsidal distances is constant, the apsidal angle increases as the second apsidal distance decreases.*

[For orbits having a common pericentral distance this means that the apsidal angle is greater the more nearly circular the orbit; for orbits having a common apocentral distance the reverse is true.]

Case I. Let $1/\beta$ be the common pericentral distance of two orbits, whose apocentral distances are $1/\alpha_1$ and $1/\alpha_2$, and whose constants of areas are h_1 and h_2, $h_2 > h_1$. Now, from the differential equations (1) of the two orbits, it follows that for equal values of u_1 and u_2

$$(7) \qquad h_2^2\left(\frac{du}{d\theta}\right)_2^2 = h_1^2\left(\frac{du}{d\theta}\right)_1^2 + (h_2^2 - h_1^2)(\beta^2 - u^2).$$

And, since $(du/d\theta)_1$ is real for $\beta \geqq u_1 \geqq \alpha_1$, the same is true of $(du/d\theta)_2$, and moreover $(du/d\theta)_2^2 > 0$ at $u_2 = \alpha_1$. Hence the interval (α_1, β) is one of real motion for orbit 2; and α_2 lies outside the interval, so that $\alpha_2 < \alpha_1$. The apsidal angles Θ_1 and Θ_2 are given by

$$(8) \qquad \Theta_i = \int_{\alpha_i}^{\beta} \frac{du_i}{\sqrt{\beta^2 - u_i^2 + \frac{1}{h_i^2}\left[F(u_i) - F(\beta)\right]}} \qquad (i = 1, 2).$$

To compare these two angles, make the following linear substitutions: $u_1 = a\beta + bu$, $u_2 = u$, where

$$a = \frac{\alpha_1 - \alpha_2}{\beta - \alpha_2}, \quad b = \frac{\beta - \alpha_1}{\beta - \alpha_2} = 1 - a.$$

Both a and b are evidently positive and less than unity; and $u_1 > u$ if $\beta > u$. Then $\Theta_1 - \Theta_2$ becomes

$$\Theta_1 - \Theta_2 = \int_{\alpha_2}^{\beta} \left\{ \frac{b}{\sqrt{\beta^2 - (a\beta + bu)^2 + \frac{1}{h_1^2}[F(a\beta + bu) - F(\beta)]}} - \frac{1}{\sqrt{\beta^2 - u^2 + \frac{1}{h_2^2}[F(u) - F(\beta)]}} \right\} du, \tag{9}$$

for $u = \alpha_2$ at $u_1 = \alpha_1$. A function similar to that used in proving Theorem I will be defined here,

$$(10) \quad \Psi(u) \equiv h_1^2 b^2 [h_2^2(\beta^2 - u^2) + F(u) - F(\beta)] - h_2^2[h_1^2(\beta^2 - (a\beta + bu)^2) + F(a\beta + bu) - F(\beta)].$$

Since in both orbits $du/d\theta$ vanishes for $u = \alpha_2$ and for $u = \beta$,

$$(11) \qquad \Psi(\alpha_2) = \Psi(\beta) = 0.$$

Evidently

$$\begin{aligned} b^{-2}\Psi''(u) &= h_1^2 F''(u) - h_2^2 F''(a\beta + bu) \\ &= 2h_1^2 P'(u) - 2h_2^2 P'(a\beta + bu) \end{aligned}$$

or

$$(12) \quad \tfrac{1}{2} b^{-2}\Psi''(u) = (h_1^2 - h_2^2)P'(u) - h_2^2[P'(u_1) - P'(u)].$$

Since $h_2 > h_1$ and $P'(u)$ is positive, the first term on the right side of (12) is negative, and since $P'(u)$ increases with u or is constant, and $u_1 > u$ for $\beta > u$, the second term is negative or zero. Hence $\Psi''(u) < 0$ for $u < \beta$, and $\Psi'(u)$ decreases from $u = \alpha_2$ to $u = \beta$, vanishing at most once, because continuous. But, from (11), it vanishes once; this shows that $\Psi'(\alpha_2) > 0$ and $\Psi'(\beta) < 0$. Consequently $\Psi(u) > 0$ for $\beta > u > \alpha_2$, and the integrand is positive throughout, showing that $\Theta_1 - \Theta_2 > 0$.

Now it was shown that $h_2 > h_1$ implies $\alpha_2 < \alpha_1$; also it is obvious that $h_2 = h_1$ implies $\alpha_2 = \alpha_1$. It follows, therefore, that the greater the value of α — that is, the more nearly circular the orbit — the greater is the apsidal angle. The theorem is thus established for families of orbits with the same pericentral distance; it remains to examine those having the apocentral distance constant.

Case II. Let $1/\alpha$ be the common apocentral distance of two orbits whose pericentral distances are $1/\beta_1$ and $1/\beta_2$, and whose constants of areas are h_1 and h_2, $h_2 > h_1$. An examination of the differential equations corresponding to (1) shows that for equal values of u_1 and u_2

$$(13) \qquad h_1^2\left(\frac{du}{d\theta}\right)_1^2 = h_2^2\left(\frac{du}{d\theta}\right)_2^2 + (h_1^2 - h_2^2)(\alpha^2 - u^2).$$

And, since $(du/d\theta)_2$ is real for $\beta_2 \geqq u_2 \geqq \alpha$, the same is true of $(du/d\theta)_1$, and moreover $(du/d\theta)_1^2 > 0$ at $u_1 = \beta_2$. Hence the orbit 1 is real throughout the interval (α, β_2), and beyond. Therefore $\beta_1 > \beta_2$. The difference of the apsidal angles may then be written

$$(14) \qquad \begin{aligned} \Theta_1 - \Theta_2 = \int_\alpha^{\beta_1} \Bigg\{ & \frac{1}{\sqrt{\alpha^2 - u^2 + \frac{1}{h_1^2}\left[F(u) - F(\alpha)\right]}} \\ & - \frac{f}{\sqrt{\alpha^2 - (e\alpha + fu)^2 + \frac{1}{h_2^2}\left[F(e\alpha + fu) - F(\alpha)\right]}} \Bigg\} du, \end{aligned}$$

where

$$e = \frac{\beta_1 - \beta_2}{\beta_1 - \alpha}, \quad f = \frac{\beta_2 - \alpha}{\beta_1 - \alpha} = 1 - e.$$

Evidently both e and f are positive and less than unity; and $e\alpha + fu < u$ for $\alpha < u$. Let

$$(15) \qquad \begin{aligned} \chi(u) \equiv h_1^2\left[h_2^2(\alpha^2 - (e\alpha + fu)^2) + F(e\alpha + fu) - F(\alpha)\right] & \\ - f^2 h_2^2\left[h_1^2(\alpha^2 - u^2) + F(u) - F(\alpha)\right]. & \end{aligned}$$

Then, for the same reason as in (11),

$$(16) \qquad \chi(\alpha) = \chi(\beta_1) = 0;$$

and

$$(17) \quad \frac{1}{2f^2}\chi''(u) = (h_1^2 - h_2^2)P'(u) - h_1^2[P'(u) - P'(e\alpha + fu)].$$

Since both terms of the right member of (17) are negative for $u > \alpha$, $\chi''(u) < 0$ so that the discussion of $\Psi(u)$ in Case I applies equally to $\chi(u)$ here. Hence, $\Theta_1 - \Theta_2 > 0$, or $\Theta_1 > \Theta_2$. Finally, then, it follows that the smaller the pericentral distance — that is, the less nearly circular the orbit — the greater the apsidal angle.

Corollary. *If two orbits have no apsidal distance in common, both apsidal distances in the first being less than the corresponding distances in the second, the apsidal angle in the first is greater than that in the second.*

For, let the apocentral and pericentral distances in orbit 1 be, respectively, $1/\alpha_1$ and $1/\beta_1$, and in orbit 2, $1/\alpha_2$ and $1/\beta_2$. Consider a new orbit — orbit 3 — with pericentral distance $1/\beta_1$ and apocentral distance $1/\alpha_2$. Then, by Case I, $\Theta_1 > \Theta_3$; and by Case II, $\Theta_3 > \Theta_2$. Hence, $\Theta_1 > \Theta_2$.

Theorem III. *If $P'(u)$ is negative, and a decreasing function of u or constant, then in those families of orbits one of whose apsidal distances is constant, the apsidal angle increases with the other apsidal distance.*

The proof is got as in Theorem II, except that $\Psi''(u)$ and $\chi''(u)$ are both positive, which reverses the sign of the integrand in each case. Hence the conclusion. The reverse of the above corollary holds true here.

[Remark. Theorems II and III might have been anticipated intuitionally from the fact that the difference between the variations of the given law and that of Newton is most marked near the center of force. One should therefore expect the apsidal angle to differ from π by larger amounts, the nearer the center the orbit extends.]

Theorem IV. *About the center of force there exists a system of concentric spheres, such that those orbits lying wholly between the surfaces of consecutive spheres have their apsidal angles all greater than π, or else all less than π.*

[The hypothesis upon $P(u)$, viz., $P'(u)$ finite save at a finite number of points in any interval, is, however, to be strengthened: $P(u)$ has only a finite number of maxima and minima in any interval.]

By reason of these two hypotheses there exist, in any interval, only a finite number of values of r for which $P'(u) = 0$, or $P'(u) = \infty$. Select these values as radii of concentric spheres. Then, between consecutive surfaces, $P'(u)$ neither vanishes nor becomes infinite, and consequently keeps the same sign. Hence, the difference $\Theta - \pi$ has the same sign for all orbits in the region specified.

APPLICATIONS OF THEOREMS.

While all of the above theorems will be utilized in what follows, Theorem I, by reason of its simplicity and generality, offers the most powerful criterion. As will be apparent in the second class of applications to be made, an inspection of the law of force is all that is necessary to ascertain whether the apsidal angle is greater than π in some orbits, and less than π in others, or whether the difference $\Theta - \pi$ must always have the same sign.

Class I: Motion in the equatorial plane of a spheroid.

Case I. The oblate spheroid. The resultant force varies, not inversely as the square of the distance, but in a much more complicated manner. Denoting by e the eccentricity and by a the major semiaxis of a meridian section, the components of attraction, X and Y, parallel to rectangular axes in the equatorial plane are given by *

$$(18)\quad \begin{aligned} &\frac{X}{x} = \frac{Y}{y} \\ &= -\tfrac{3}{2}\frac{k^2M\cdot m}{(ae)^3}\left[\arcsin\frac{ae}{\sqrt{x^2+y^2}} - \frac{ae}{\sqrt{x^2+y^2}}\sqrt{1-\frac{a^2e^2}{x^2+y^2}}\right], \end{aligned}$$

m being the mass of the attracted particle, M that of the spheroid, and x and y the coordinates of the particle, the origin being at the center of the spheroid. Now (18) shows that the force is central; and, introducing the variable u, it is found that

$$(19)\qquad P(u) = \frac{K}{u^3}\left[\arcsin aeu - aeu\sqrt{1-(aeu)^2}\right],$$

where K is a positive constant. The right member of (19) is representable by a power series in aeu, convergent for all points exterior to the spheroid; thus

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* Moulton, F. R. Introduction to celestial mechanics, pp. 123 ; 118.

$$(20)\qquad P(u)=\sum_{n=0}^{n=\infty} A_n(aeu)^{2n},$$

where the A_n *are all positive.* Evidently, then, for all exterior points, $P'(u)$ is positive and increases with u; consequently Theorems I and II apply.

Hence, *every orbit described in the equatorial plane of an attracting oblate spheroid has its apsidal angle greater than* π; *and, the nearer the center of force the orbit lies, the greater the apsidal angle.*

This finds an immediate application in the motion of the fifth satellite of Jupiter, whose orbit deviates exceedingly little from the equatorial plane of the planet. The oblateness of Jupiter is considerable, so that $P(u)$ differs from the newtonian term in the series (20) very markedly for large values of u. This, with the fact that the satellite's orbit is very near the planet, should explain the large difference between its apsidal angle and π, amounting in a year to about 900°.

This "advance of the line of apsides" has also been fully explained from the standpoint of perturbations,* the orbit being regarded as an ellipse whose elements vary; the difference between the apsidal angle and π being considered as a forward rotation of the major axis.

Case II. The prolate spheroid. The attraction upon a particle in the equatorial plane is given by †

$$(21)\quad P(u)=\frac{K}{u^3}\left[aeu\sqrt{1+(aeu)^2}-\log\left(aeu+\sqrt{1+(aeu)^2}\right)\right],$$

K being a positive constant, and a and e having the same meanings as in Case I. The right member of (21) is developable as a power series in aeu,

$$(22)\qquad P(u)=\sum_{n=0}^{n=\infty} B_n(aeu)^{2n},$$

where

$$B_n=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\frac{(-1)^n}{2n+3}.$$

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* Moulton, loc. cit., pp. 232–233.

† Moulton, loc. cit., p. 123.

This series is convergent for $aeu < 1$; whether this includes all exterior points depends upon the value of e.

Since the first coefficient of $P'(u)$ is negative, it is evident that $P'(u) < 0$ for sufficiently small values of u. Hence all orbits lying outside of a certain circle have their apsidal angles less than π.

Again, since the ratio of consecutive coefficients in $P''(u)$ is

$$\frac{(2n+1)^2(2n+3)}{(2n-1)(2n)(2n+5)}(-1),$$

which has its greatest numerical value, $\frac{45}{14}$, for $n = 1$, it follows that for $aeu < \sqrt{\frac{14}{45}}$, if the terms of $P''(u)$ be grouped in pairs, each pair will have a negative sum. So that, for u sufficiently small, $P'(u)$ is a decreasing function; therefore the apsidal angle decreases toward the center of force.

Finally, calling $aeu = \lambda$,

$$P'\left(\frac{\lambda}{ae}\right) = \frac{3K(ae)^3}{\lambda^4\sqrt{1+\lambda^2}}\left[\log\left(\lambda+\sqrt{\lambda^2+1}\right) - \left(\lambda + \tfrac{1}{3}\lambda^3\right)\right],$$

which can be shown to be everywhere negative. Thus, let $\log(\lambda + \sqrt{\lambda^2+1}) - (\lambda + \frac{1}{3}\lambda^3) = \Phi(\lambda)$. Then

$$\Phi(0) = 0,\ \Phi'(\lambda) = \frac{1}{\sqrt{1+\lambda^2}} - (1+\lambda^2).$$

Since $\Phi'(\lambda) < 0$ for every real value of λ, $\Phi(\lambda) < 0$ for $\lambda > 0$. Hence, *the apsidal angle is less than π in all the trajectories.*

Class II. Orbits for various simple laws.

The criterion of Theorem I yields especially simple results when applied to the following types of laws of force:

Case I. Force varies inversely as the n^{th} power of the distance. If

$$u^2P(u) = k^2\frac{1}{r^n},\quad P(u) = k^2u^{n-2},$$

and $P'(u) = (n-2)k^2u^{n-3}$. Three cases arise:

(1) $n = 2$; Newton's law, $\Theta = \pi$.

(2) $n > 2$; $P'(u) > 0$, $P''(u) \geqq 0$. Hence, theorems I and II both apply. $\Theta > \pi$, and increases toward the center.

(3) $n < 2$; $P'(u) < 0$, $P''(u) > 0$. Theorem I applies, but not Theorem III. $\Theta < \pi$.

In many of the cases included here it is quite impossible to evaluate in closed form the definite integral defining Θ. Thus, under (2) are included the well known "inverse fifth power," which leads to elliptic integrals; also, all the inverse higher powers leading to abelian and hyperabelian integrals. By Theorem I all these integrals are greater than π and increase with the apsidal values of u.

Under (3) are included: The "inverse first power," which leads to logarithms, and which is usually omitted from the treatises; the simple law of the "direct first power," where $\Theta = \frac{1}{2}\pi$; the case of the constant force, leading to an elliptic integral, and the higher direct powers with their involved integrals. These apsidal angles are, however, by Theorem I, all less than π.

Case II. *Force varies as log* $1/r^m$ *and inversely as* r^n. Here, since

$$u^2P(u) = (k^2m/r^n)\log u, P'(u) = k^2m[(n-2)\log u \cdot u^{n-3} + u^{n-3}],$$

or

$$P'(u) = k^2mu^{n-3}[1 + (n-2)\log u]. \tag{23}$$

From (23) it is evident that for $n = 2$, $P'(u) > 0$ for $u > 0$; for $n > 2$, $P'(u) \gtrless 0$ according as $\log u \gtrless -1/(n-2)$, and for $n < 2$, $P'(u) \gtrless 0$, according as $\log u \lesseqgtr 1/(2-n)$. That is, if $n > 2$, $\Theta > \pi$ in orbits within a certain sphere; if $n = 2$, $\Theta > \pi$ in all orbits; if $n < 2$, $\Theta > \pi$ in orbits without a certain sphere. The opposite inequalities hold in other parts of space.

Case III. *Force varies as* $e^{\phi(u)}$ *and inversely as* r^n. In this case, since $P'(u) = k^2u^{n-2}[\phi'(u) + (n-2)u]e^{\phi(u)}$, it follows that if $\phi(u)$ be everywhere real, $P'(u)$ has the same sign as $\phi'(u) + (n-2)u$. While a discussion of the cases arising when $\phi(u)$ is arbitrary is of small importance, the following special cases may well be noted. If $\phi'(u) > 0$ for $u > 0$, the inequalities of Case II hold; if $\phi'(u)$ is everywhere negative, those inequalities are to be reversed.

The laws of force which have been examined, while only a few, embrace most of those commonly considered in treatments of central force problems, and the simplicity of the criterion in these cases encourages the hope that it may be found useful elsewhere.

WILLIAMS COLLEGE,
May, 1907.

THE MAXIMUM VALUE OF A DETERMINANT.

BY PROFESSOR ELLERY W. DAVIS.

HADAMARD * has shown that the maximum value reached by the modulus of a determinant of order n the moduli of whose elements do not exceed unity is $n^{n/2}$. The result is fundamental in Fredholm's theory of integral equations.†

The problem, when the elements are required to be real, was studied by Professor Hathaway and myself in 1882. I do not think Mr. Hathaway's work was published. My own was published ‡ only in the abstract here reproduced.

"The elements of a determinant being restricted to a variation between the limits $-a$ and $+a$, it is found that for all determinants whose order is greater than 2, a numerical maximum will be obtained by putting the elements $-a$ in the principal diagonal and making all the other elements of the determinant $+a$. If we denote such a determinant of order n when $a=1$ by D_n and the minor of an element in the ith row and kth column by A_{ik}, we have always $A_{ii}=D_{n-1}$, $A_{ik}=-D_{n-1}/(n-3)$, so that

$$D_n = -\left(1+\frac{n-1}{n-3}\right)\cdot D_{n-1} = +\left(1+\frac{n-1}{n-3}\right)\left(1+\frac{n-2}{n-4}\right)D_{n-2}$$

$$= \pm\left(1+\frac{n-1}{n-3}\right)\left(1+\frac{n-2}{n-4}\right)\cdots\left(1+\frac{5}{3}\right)\left(1+\frac{4}{2}\right)\left(1+\frac{3}{1}\right)D_3$$

$$= \pm(n-2)2^{n-1}, \text{ since } D_3 = 4.$$

"The sign will be $\pm$ according as n is $\substack{\text{odd}\\ \text{even}}$. The effect of a change in any element is to lessen the greatest factor leaving the rest the same.

"For the maximum cubic determinant $D_n^{(3)}a^n$, we have merely to make all the strata identical and equal to $D_n^{(2)}$. Its value is $\pm n!\,D_n^{(2)}a^n$.

"The four-dimensional determinant may be looked upon as a sort of determinant of plane determinants, the terms of the devel-

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* *Bulletin des sciences math.*, 1893, p. 240. Pascal, I Determinanti, p. 240.

† *Acta Math.*, vol. 27 (1903), p. 365.

‡ Johns Hopkins University Circular, vol. 2, No. 20 (December, 1882), p. 22.

opment being cubic determinants of which the plane determinants are the strata. In this quasi-determinant the same rule of signs applies as in the ordinary determinant and we get for the maximum value accordingly

$$\pm D_n^{(2)} n\,!\, D_n^{(2)} a^n = \pm\, n\,!\, (D_n^2)^2.$$

Similarly

$$D_n^{(2p)} = \pm\, (n\,!)^{p-1} (D_n^{(2)})^p, \qquad D_n^{(2p+1)} = \pm\, (n\,!\, D_n^{(2)})^p.$$

"In any of these determinants we have at once a formula for the maximum number of positive terms in the development. This is for $D_n^{(q)}$

$$\frac{(n\,!)^{q-1} \pm D_n^{(q)}}{2}.$$

"We use the ± sign according as $D_n^{(q)}$ is ±."

A simple example serves to show that the limit reached when the elements are real may be exceeded when the elements are complex. Thus

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4,$$

but

$$\begin{vmatrix} i & 1 & 1 \\ 1 & \dfrac{-1+i}{\sqrt{2}} & 1 \\ 1 & \dfrac{1+i}{\sqrt{2}} & -1 \end{vmatrix} = 2(\sqrt{2}+1).$$

This explains why Hadamard's limit exceeds mine.

UNIVERSITY OF NEBRASKA,
July, 1907.

THE INVARIANT SUBSTITUTIONS UNDER A SUBSTITUTION GROUP.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, September 6, 1907.)

We begin with the case where the substitution group G is transitive and of degree n. If the subgroup G_1 composed of all the substitutions of G which omit a given letter is of degree $n-1$, there is no substitution which involves any of the letters contained in the substitutions of G and is also commutative with every substitution of G. In considering substitutions which are commutative with every substitution of G we shall confine ourselves to those which involve no letters that are not also contained in G, since every substitution which does not involve any of the letters of G is clearly commutative with every substitution of G. Hence we may say: *when the degree of* G_1 *is* $n-1$, *identity is the only substitution which is commutative with every substitution of* G.

When G_1 is identity it is well known that there are exactly n substitutions which are commutative with every substitution of G, and that these constitute a group which is simply isomorphic with G, known as the *associate* of G, whenever G is non-abelian.* It remains to consider the case where G_1 is of degree $n-\alpha$ $(1<\alpha<n)$. All the substitutions which transform G_1 into itself constitute a subgroup of order $\alpha g \div n$, g being the order of G. This subgroup may clearly be constructed by establishing a $(g_1, 1)$ isomorphism between a group of order $\alpha g \div n$ and a regular group of order α, g_1 being the order of G_1. If this regular group is abelian it is transformed into $n \div \alpha$ distinct regular groups by the substitutions of G and a simple isomorphism between these groups which is so constructed that all the conjugates of a given substitution correspond will be composed of all the substitutions which are commutative with every substitution of G. If the regular group of order α is non-abelian its associate is transformed in the manner stated. Hence we have the theorems: *The necessary and sufficient condition that there are substitutions besides identity which are com-*

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* *Quar. Jour. of Math.* vol. 28 (1896), p. 249. When G is abelian it is self-associate.

mutative with every substitution of a transitive group of degree n is that the subgroup which is composed of all its substitutions omitting a given letter is of a degree lower than $n - 1$. If the degree of this subgroup is $n - \alpha$, there are exactly α substitutions which are separately commutative with every substitution of the transitive group, where α may be any divisor of n but can have no other value.

This theorem may be regarded as a generalization of the theorem that there are just n substitutions which are commutative with every substitution of a regular group. It is also directly applicable to intransitive groups, since the total number of substitutions which are commutative with every substitution of such a group and do not interchange any of its systems of intransitivity is composed of the direct product of those which are commutative with all the substitutions of its transitive constituents. While the number of substitutions which are commutative with every substitution of a transitive group is always a divisor of the degree of the group, the number of those which have this property and do not permute the systems is a divisor of the product of the partitions of this degree with respect to addition when the group is intransitive.* Although every transitive group which is such that there are substitutions which are commutative with every one of its substitutions is necessarily imprimitive;† it is not true that every imprimitive group has this property, since there are imprimitive groups of degree n which involve subgroups of degree $n - 1$. The preceding developments have close contact with those given on page 124 of Maillet's Thèse de Doctorat, 1892, and by Burnside, "Theory of groups of finite order," 1897, page 217.

The α substitutions which are commutative with every substitution of G when G_1 is of degree $n - \alpha$ are not necessarily contained in G whenever $\alpha > 1$. They are however contained in the largest group of degree n which transforms G into itself. We thus arrive at a generalization of a property of the holomorph of a group, which may be stated as follows: The largest group of degree n which transforms a transitive group of degree

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* The addends are clearly the degrees of the systems of intransitivity and the total number of substitutions which are commutative with every substitution of an intransitive group is a divisor of the product of the degrees of its transitive constituents and the number of ways in which these constituents may be transformed into each other.

† Except the cyclic group of prime order.

n into itself contains α substitutions which are commutative with every substitution of this transitive group when its subgroup composed of all the substitutions which omit a given letter is of degree $n - \alpha$. In the case of the holomorph of G, $\alpha = n$.

UNIVERSITY OF ILLINOIS,
July, 1907.

SHORTER NOTICES.

Leçons d'Algèbre et d'Analyse à l'Usage des Elèves des Classes de Mathématiques spéciales. Par JULES TANNERY. Tome Premier. Paris, Gauthier-Villars, 1906. 423 + vii pp.

THIS text is prepared for students of the classe de mathématiques spéciales of the French lycées, in which boys of 18 or 19 prepare themselves for admission to the Ecoles normale and polytechnique. In so doing, they go over an amount of mathematics which seems overwhelming.* If the contents of the volume under review are even approximately an indication of what is taught successfully in one year, in any one of several subjects pursued by classes of boys of ordinary ability, then we have indeed much to learn from the teachers in the French lycées. But the contrast is less pronounced when we consider the difference in aim. The French adapt their courses to the abilities of the 2 per cent. or 5 per cent. who are the most gifted; Americans adapt their work to the capacity of the average boy. In the French lycées the 95 per cent. or 98 per cent. who fail in the first trial repeat the course during a second year, whereupon about 25 per cent. succeed. Less than half are said to succeed even after the third trial. Much can be said for and much against such a system of highly competitive examinations.

The first chapter of Tannery's book, covering 58 pages, contains a detailed exposition of irrational numbers. Adopting an easy conversational style, the author makes the subject very clear, except that, in a few instances, it is not stated from the start what the underlying assumptions are. Whenever such doubt lurks in the mind of the reader, he will find that the

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* BULLETIN, vol. 6 (1900), p. 233.

P 35,679

matter is cleared up later on, when a restatement of the argument is made. Thus, on page 6, the author states in a footnote the assumption, now usually called the assumption of Archimedes, which was tacitly made on page 4. Tannery calls irrational numbers (page 2), "une façon de parler plutôt qu'une réalité." His theory of the irrational, although originally developed independently, resembles that of Dedekind in starting out with the notion of a partition (coupure). In the elegant development of this theory in the volume under review, the author makes ingenious use of a color system. Of the two classes into which a segment incommensurable to the unit segment on a line divides the points standing for rational numbers, he imagines the inferior class to be represented by blue points and the superior class by red points; while the irrational number in question is pictured by a white point. When the partition is effected by a commensurable segment, the points representing rational numbers are divided into three classes, namely the inferior (blue), the superior (red), and the single intermediate (white). This color imagery greatly facilitates the discussion.

Thus, two irrational numbers A and B are said to be equal when the blue and red points of the one are the very same, respectively, as those of the other. The sum (product) of A and B is a number greater than the sum (product) of any two blue numbers belonging, respectively, to A and B; this sum (product) is a number less than the sum (product) of any two red numbers.

Chapters II and III are given to polynomials. The real object of algebra is asserted to be the study of polynomials from different points of view. The author explains the graphic representation and the differentiation of polynomials, derives Taylor's theorem by the method of undetermined coefficients, and touches upon the singular points of graphs.

In chapters IV–VII are treated rational fractions, the greatest common divisor, and imaginaries. The resolution into partial fractions is effected by a process unusual in our text-books. Suppose $f(x)$ is a polynomial of lower degree than $\phi(x) \equiv (x-a)^\alpha(x-b)^\beta \cdots (x-l)^\lambda$. Writing $x = a + h$, $\phi_1(x) \equiv (x-b)^\beta \cdots (x-l)^\lambda$, and

$$P_\alpha\left(\frac{1}{x-a}\right) \equiv \frac{A_0}{(x-a)^\alpha} + \frac{A_1}{(x-a)^{\alpha-1}} + \cdots + \frac{A_{\alpha-1}}{x-a},$$

we obtain

$$\frac{f(a+h)}{\phi_1(a+h)} = A_0 + A_1h + \cdots + A_{\alpha-1}h^{\alpha-1}\frac{h^\alpha R(h)}{\phi_1(a+h)}.$$

Dividing by h^α and then writing $x - a$ for h, we have

$$\frac{f(x)}{\phi(x)} = P_\alpha\left(\frac{1}{x-a}\right) + \frac{f_1(x)}{\phi_1(x)}.$$

This process is repeated on $f_1(x)/\phi_1(x)$ with respect to $(x - b)^\beta$, and so on. It is thus made evident in a natural and convincing way that, for any factor $(x - a)^\alpha$ of the denominator, all powers of $x - a$, from 1 to α, must in general appear as denominators of partial fractions. The numerators A_0, A_1 $\cdots$ are determined by the process of division.

In the treatment of imaginaries Tannery is influenced by Cauchy, who looked upon i as a real variable. The "fundamental theorem of algebra" is expressed by Tannery on page 223 in this form: Given a polynomial with real coefficients in two variables x and i, then there is a binomial $\alpha + \beta i$ (α and β real) such that, on writing $\alpha + i\beta$ for x, the result is divisible by $i^2 + 1$. The letter i plays only a very special role; for, the remainder resulting from a division of a polynomial by $i^2 + 1$ is the sole object of consideration. Two polynomials involving i are said to be equal when their remainders are equal, or when their difference is exactly divisible by $i^2 + 1$. Tannery remarks that this definition satisfies the three essential conditions of equality: an object is equal to itself; if a first object is equal to a second, then the second is equal to the first; if two objects are equal to a third, they are equal to each other. It is then evident that a polynomial in i is equal to its remainder, and in particular that the polynomial i^2 is equal to -1, which is the remainder resulting from the division of i^2 by $i^2 + 1$. When no confusion arises, one may regard i either as a real variable or as defined by $i^2 = -1$. From either point of view, two complex numbers $a + a'i$ and $b + b'i$ are equal if $a = b$ and $a' = b'$. The definitions of the fundamental operations on imaginaries are then set forth.

In the graphic representation of imaginaries the author starts out by marking off real and pure imaginary numbers along rectangular axes, without attempting to assign any reason for avoiding oblique axes. Later a justification for this procedure appears as the result of successive multiplication of the vector 1 by i. The chapter ends with a trigonometric treatment of the roots of unity.

The remaining three chapters are on combinations and permutations, the binomial formula, equations of the first degree and determinants. Everywhere we recognize the resolute endeavor to present the subject with perfect sincerity. The author says in the preface: "J' ai horreur d'un enseignement qui n'est pas toujours sincère: le respect de la vérité est la première leçon morale, sinon la seule, qu'on puisse tirer de l'étude des sciences." F. CAJORI.

Leçons d'Algèbre et d'Analyse. Par JULES TANNERY. Tome Second. Paris, Gauthier-Villars, 1906. 638 pp.

THIS volume, which is the second of a comprehensive treatise on this field of algebra and analysis, like the other volumes, is prepared for the use of special students of mathematics of the Sorbonne, Paris. The chapter headings are as follows: series, functions of a real variable, series of functions, applications in the study of a function in the separation and calculation of the roots of an equation, algebraic equations, differential notation and plane curves, and notions of integral calculus.

The chapter on series deduces, with a simplicity and clearness that are ample for beginners of collegiate grade, the fundamental notions of the subject, establishes the ordinary tests for convergence and divergence, examines for convergence many series of frequent occurrence, and closes with seven pages of interesting exercises on convergence, divergence, and equivalence of series. All this is done in the compass of 53 pages and that, too, without material omission of essentials.

The meanings of variable, of function, of the phrases "appertaining to an interval" and "lying within an interval," and of bounds are given and exemplified in the next chapter on functions of a real variable. Common geometric notions are first given, then shown to be too indefinite for the purposes of analysis; then the analytic definitions are given for the following: curve, continuity, functionality, and increasing, decreasing, and discontinuous functions. The meaning and domains of validity of inverse, logarithmic, circular, and exponential functions are pointed out. A development of the properties of hyperbolic functions completes the theoretical part of the chapter. The principles deduced are reviewed and applied by a four-page list of well-selected exercises.

The third chapter renders precise and trustworthy the terminology and basic notions involved in deriving functions of real variables, and gives general methods of deriving, both totally and partially, functions of real variables with real, and with imaginary coefficients. Rolle's theorem and the so-called theorem of finite differences are deduced and four pages of exercises are added. The functions here treated being capable of graphic representation, the author satisfies himself with the degree of rigor which such functions require, and postpones for later consideration the restrictions necessary for absolute analytical rigor. At appropriate places the student is cautioned as to the inadequacy of these first treatments. He is thus led to feel that while he is always getting the truth, and as much of it as he can assimilate, yet it is not the *whole* truth. This procedure cultivates in the student open-mindedness and the feeling that there is still more to learn.

Series whose terms are functions of a variable, series in integral powers of a single variable—power series—first with real, then with imaginary coefficients, and the Taylor and Maclaurin developments are carefully treated and extensively applied to both algebraic and transcendental functions in the next chapter. Several illusory forms are given special attention and their true values are defined.

Then follows a study of variations of functions, of roots, multiple roots, asymptotes, convexity, concavity, points of inflexion, and separation of roots. Rolle's theorem is again taken up, and methods of approximating roots of functions are taught and given extensive application.

The relations of coefficients and of roots of algebraic equations, symmetric functions, Waring's method, methods of elimination and transformation, reciprocal, binomial, cubic, and quartic equations, together with other topics of the theory of algebraic equations, are given full and yet concise consideration. The differential and determinant notations applied to the study of plane curves, with a clearly logical and lucid development of the integral calculus, running through 154 pages, constitute the next chapter, which completes the volume.

From this synopsis it is plain that the French order of procedure—for M. Tannery's order is customary in France—

through the several branches of mathematical analysis differs materially from the order of American texts. Of course, it does not follow that because another nation prefers a certain plan it is the best thing for us to adopt the plan. It must be admitted, however, that the loosely articulated subject matter as given in our so-called college algebra, which closes with a rudimentary and isolated treatment of the theory of equations, then trigonometry, then analytics, then calculus, leaves much to be desired. Many desirable things might be suggested by the close study of such masterly treatments as this of M. Tannery by would-be authors of college algebras and of other mathematical books for use in our schools. Both in point of gradation of inherent difficulties and in point of simplicity or treatment without loss of rigor the French text-book writers can teach us much. The logical order and mode of this volume are admirable.

Again, the plan by which M. Tannery gradually inducts the learner into the logical niceties of the subject, by giving him from time to time enough to satisfy present needs and as much as his stage of mathematical maturity and advance justifies, then reverting to the matter later, when experience has ripened sufficiently to enable the novice to interpret what he is doing, is a distinct advance pedagogically upon the common practice of American college professors. The latter are too prone to seek fulness of detail the first time the subject is taken up. The reviewer would invite their attention to M. Tannery's procedure, for these Leçons were evidently prepared to inform and to educate rather than merely to present a beautifully perfect mathematical exhibit.

Only half a dozen typographical errors, and these of no particular consequence, have been discovered in the entire volume. Typography and general mechanical excellence are sufficiently guaranteed by the imprint of Gauthier-Villars.

G. W. MYERS.

Mathématiques. Principes et Formules de Trigonométrie Rectiligne et Sphérique. Par J. PIONCHON, Professeur à la Faculté des Sciences, Directeur de l'Institut Electrotechnique de l'Université de Grenoble. Paris, Gauthier-Villars, et Grenoble, A. Gratier et J. Rey, 1904. 5 francs.

THIS is one of the 70 odd volumes now appearing from the press of Gratier et J. Rey, of Grenoble, under the editorial supervision of J. Pionchon and bearing the general title Bibliothèque de l'Elève-Ingenieur. This Bibliothèque is to consist of five sections entitled respectively: Mathématiques, Mécanique, Physique industrielle, Electricité industrielle, and Economie industrielle. The collection as a whole is to comprise a recapitulation of the fundamental notions, both theoretical and practical, of the applied sciences. M. Pionchon is to be assisted in this undertaking by about a dozen of the foremost representatives in France of the several subjects to be treated. The book under review is one of the half-dozen numbers of this collection that have recently appeared. A dozen other volumes of the general series are announced to be in preparation.

This little volume of 146 pages is designed for use as a hand-book for practical engineers. It is contended by the author that the needs of the practicing engineer for trigonometry are met neither by the standard texts of trigonometry prepared by professors for their classes, nor by the treatises written by experts for mathematicians. The former are held to be too scanty and the latter too extensive for practice. It is believed therefore that in the bibliography for engineering students there is a real want which this compilation seeks to supply.

This book, therefore, is not intended for a first study of the subject. Nor is it a mere compilation of the formulas needed in engineering practice. It is a sort of running sketch of the substance of the theory of trigonometry, of sufficient fulness to enable one who has once learned the science to readily recover the *rationale* of formulas and transformations. A feature of the book is the large number of such approximate formulas as are useful in practice. Lists of trigonometric integrals and of trigonometric series, full enough to meet the demands of extended practice, are also included. It is perhaps needless to say that special attention has been given to adaptations and simplifications of formulas for the solution of triangles, both plane and spherical.

A rather full appendix, comprising notes on Divers cases of maximum and minimum of certain geometric magnitudes in figures satisfying given conditions, is also included. In this appendix such questions are treated as "To find on a right line the smallest segment which may be seen from a given point under a given angle"; and "Through a point situated on the interior of an angle, to insert between the sides a rectilinear segment of minimum length," etc.

The pages are fairly well supplied with footnotes pointing out sources in which fuller detail of development can be seen, or drawing attention to certain special features of transformations that are worthy of particular attention by the practitioner.

On the whole, the volume is an unusually practicable summary for workers in trigonometry, is singularly free from typographical errors, and is printed in type of such size and variety as to enable the eye to catch readily the particular thing the reader may want. As a ready reference handbook it leaves little to be desired.

G. W. MYERS.

Beispiel-Sammlung zur Arithmetik und Algebra. Von Dr. Hermann Schubert, Professor an der Gelehrtenschule des Johanneums in Hamburg. Dritte, durchgesehene Auflage. Leipzig, G. J. Göschen'sche Verlagshandlung, 1905. 136 pp.

THIS is one of the little 80-pfennige handbooks of the well-known Sammlung-Göschen. On 120 pages, 3 inches by 4 inches, it contains 1275 exercises and problems, formal and clothed (eingekleidet), covering the following topics: the transition from calculation to arithmetic, modes of calculating of the first order, of the second order, applications of these modes, quadratics, modes of calculating of the third order, an appendix of problems on higher arithmetic, and selected results.

The appendix contains, in the language of problems: observations on building arithmetic systematically, arithmetic and geometric series, compound interest and annuities, the binomial theorem, Moivre's theorem, and cubic equations.

The problems are arranged in a developmental order, according to the German idea of development. In mathematics the German notion, as shown by their text-book literature, seems to be to begin a topic with a large number of formal and easy exercises, to pass by easy gradations to and through more complicated problems, still of the formal type, and, lastly, to give a list of carefully graded verbal problems having a real content. This accords with the views of some American writers on high school mathematics.

In the view of other writers in our own country and in England, we should get on much better if pupils were not first mechanized by the formal problems. These writers favor fewer of the formal type of problem, and a much larger proportion than is customary of the clothed (eingekleidet) type, the formal exercises to be put at the end of the treatment of topics, to develop mechanical skill, after a larger amount of practice is had in the translation of verbal into formal language. For a considerable time a necessary part of the solution of every problem would be the setting up of the necessary equations. The former mode seeks technique first, then undertakes to infuse thinking into a formal frame-work. The latter seeks to secure the thinking first, and then to develop the technique *as a means of facilitating*, not result-finding, but *thought*. In the reviewer's opinion this collection errs in the over-stressing of the formal side of algebra at the beginning of developments of topics. Of course, teachers may select problems in whatever order they choose. But these lists have evidently been prepared for the special service of teachers whose practice is "through the art to the science."

In this compilation the formal problems outnumber the clothed, or verbal, about in the ratio of 3 to 1. The verbal problems are taken from physical science, from geometry, from elementary mechanics, and from the customary topics of arithmetic. A very large percentage of them are of real modern interest. Any teacher of high school algebra will find this manual a valuable source from which to select exercises to replace many of the dead ones of the standard texts. In this day of correlated algebra and geometry it will be a great help to learn, as these lists of problems show, how many types and varieties of algebraic equations may be based on geometric relations. The chief value of the book for American high school teaching is its adaptability to this service. Many teachers will be glad to find so practical a means as such problems afford of holding the ground made in algebra during the first high school year while the second-year geometry is being taught.

G. W. MYERS.

An Elementary Treatise on Pure Geometry. By J. W. RUSSELL, M.A. Oxford, The Clarendon Press, 1905. xii + 366 pp.

THIS is the second edition of a work in which "the author has attempted to bring together all the well-known theorems and examples connected with harmonics, anharmonics, involution, projection (including homology), and reciprocation" (preface to first edition, line 1). In other words, it is concerned with the material of the usual courses on projective geometry, including problems of the first and second degree in the plane. It starts with a metric foundation, Menelaus's and Ceva's theorems, etc., and uses metric geometry freely throughout the book.

Some of the fundamental concepts are introduced in a way that is not likely to impress a beginning student (or any one else) with its logical sharpness. For example, at the bottom of page 9, the notion of "point at infinity" is introduced as follows:

"Take any fixed point O and a fixed line l. Then any line x through O cuts l in a point P. Now rotate x about O so that x may become more and more nearly parallel to l. Then P recedes indefinitely along l; and in the limit when x is parallel to l, P is said to be the point at infinity upon l. Hence *two parallel lines intersect in a point at infinity.*" (Author's italics.)

From here on the "point at infinity" is used quite freely. At the beginning of Chapter III occurs the first mention of imaginary points. It is as follows:

"1. EVERY *line meets a circle in two points, real, coincident, or imaginary.*

"For take any line l cutting a circle in the points A and B. Now move l parallel to itself away from the center of the circle. Then A and B approach and ultimately coincide when l touches the circle. But when l moves still further from the center, the points A and B disappear; yet, for the sake of continuity, we say that they still exist, but are *imaginary* (see also XXVII)."

The reference at the end is to Chapter 27, where the aid of algebraic geometry is invoked: "* * * this quadratic equation will have two solutions, real, coincident, or imaginary. Hence we conclude that a line always meets a circle in two points, real, coincident, or imaginary."

It may be true, as many teachers evidently believe, that clear statements on subjects like those to which our quotations apply are too difficult for most elementary students. But on the other hand must be set the distrust inspired in thoughtful students by statements which, taken literally, are untrue, and the flabbiness of mind which comes from arguing with vaguely defined terms.

These criticisms apply only to a limited number of places in Mr. Russell's book and do not in any way impeach its usefulness as a source of theorems and examples. The latter are particularly rich in metric cases of projective theorems. The arrangement of material has a number of elegant features as, for example, the way in which theorems on conic sections are derived by projection and reciprocation from the corresponding theorems on circles.

O. VEBLEN.

Wahrscheinlichkeitsrechnung und Kollektivmasslehre. By Dr. HEINRICH BRUNS. Leipzig, Teubner, 1906. 310 + 18 pp.

ABOUT one third of this book is devoted to the theory of probability, and two thirds to Kollektivmasslehre. The theory of probability is treated as a theory of frequency, and from this point of view the part on probability is presented in excellent form for application to Kollektivmasslehre. The intimate relation between the two parts of the book stands out so clearly as to make it an important feature, especially because in the work of Fechner Kollektivmasslehre appears much more as an independent subject than as one so closely related to the theory of probability.

While Fechner and Pearson have, to a certain extent, treated Gauss's law of distribution as a "scientific dogma," and have presented generalized probability curves which fit well a large class of data, the conclusion that all distributions conform to one of these curves would have the same kind of logical weakness as the dogma of Gauss. Bearing on this point, Bruns makes a distinct advance by obtaining what seems to be a "suitable" analytic representation for an arbitrary frequency distribution. I use the term "suitable" because it is not difficult to get an analytic representation whose algebraic and numerical complications make it of no value for describing populations such as arise in applications.

Starting with a frequency distribution, the author constructs what he calls a "Summentafel" which gives the number of variates below given values. He uses the term "Summenfunktion" $S(x)$ to represent the relative frequency with which a variate lies below x. This function $S(x)$ and its derivative $V(x)$ (Verteilungsfunktion) are the functions for which the author obtains analytic representations. He treats the Summenfunktion as of fundamental importance rather than the distribution function, and in the process of reduction or smoothing which he employs the Summentafel is unchanged, while the frequency distribution may be very much changed.

The analytic representation of the arbitrary Summenfunktion is primarily in terms of two functions, ϕ and R. The function $\phi(x)$ is defined by

$$\sqrt{\pi}\phi(x) = 2\int_0^x \exp.\ (-t^2)dt,$$

and the function $R(x)_q$ is given by the recursion formula

$$\exp.\ (-2xv - v^2) = \sum_{q=0}^{q=\infty} R(x)_q(2v)^q.$$

If the qth derivative of $\phi(x)$ is denoted by $\phi(x)_q$ the function $S(y)$ is capable of representation in the form

$$2S(y) - 1 = \sum_{q=0}^{q=\infty} \mathfrak{D}[R(u)_q]\phi(v)_q,$$

where

$$u = h(x-c), \quad v = h(y-c)$$

and the operation $\mathfrak{D}$ is defined, in general, by

$$\mathfrak{D}[T(x)] = \int T(x)V(x)dx,$$

where $V(x)$ is the distribution function.

The c and h are parameters so chosen that

$$c = \mathfrak{D}(x), \quad h^2 = \frac{1}{2\mathfrak{D}[x-c]^2}.$$

When thus selected, c is the arithmetic mean and h is the measure of precision.

It is next shown that the above analytic representation gives

$$2S(x) - 1 = \sum_{q=0}^{q=\infty} D(c, h)_q\phi(u)_q.$$

On choosing c and h as just explained,

$$D(c, h)_1 = 0, \quad D(c, h)_2 = 0.$$

Then

$$2S(x) - 1 = \phi(u) + D_3\phi(u)_3 + D_4\phi(u) + \cdots.$$

If the D_3, D_4, $\cdots$ are zero or so small that they can be neglected, the Verteilungsfunktion gives the simple exponential law. According to this, the author says that populations can be divided into two large classes: "the first embraces the forms for which the exponential law may be regarded as at least a first approximation; the second, on the contrary, embraces the forms for which this law can not be regarded as even a crude approximation. Thus, the exponential formula, which for a long time was conceived as a sort of law of nature, is put in its proper light. For a large group of distributions arising in Kollektivmasslehre it plays a role similar, say, to the representation of the earth by means of a sphere in measurements of the earth."

In the chapter on the mixture of the arguments, it is shown that this process produces a tendency towards the exponential law. This result is of significance as accounting in part for the large number of distributions which approach this law. In the chapter on the mixture of distributions the essentials of correlation theory are treated.

Of the twenty-four chapters contained in the book, the last five deal with numerical applications. These numerical cases show the systematic methods of carrying the theory into practice, and indicate what parts of the work can be done once for all in many applications.

Dealing with a subject which we should like to see treated rigorously, this book takes a high place in point of mathematical elegance, and it should serve to make much better known this important field of applied mathematics.

H. L. RIETZ.

Hermann Grassmanns gesammelte mathematische und physikalische Werke. Herausgegeben von FRIEDRICH ENGEL. Bd. 2: Theil 1, *Die Abhandlungen zur Geometrie und Analysis*, x + 451 pp., 1904; Theil 2, *Die Abhandlungen zur Mechanik und zur mathematischen Physik*, viii + 266 pp., 1902. Leipzig, B. G. Teubner.

THE first volume of Grassmann's works is in two parts, each containing one of the two Ausdehnungslehren; the second volume reprints the miscellaneous papers and Nachlass. It is interesting to note that the earliest paper is a "Programm" on crystals and bears the inscription: Stettin, 1839. This was only five years before the publication of the first Ausdehnungslehre. The last papers are dated 1877 to 1879 and are concerned with various applications of the calculus so intimately associated with the name of the author. As a matter of fact practically all the intervening matter deals with some phase of mathematics closely allied to one or the other of the Ausdehnungslehren. The author's whole mathematical activity may therefore be said to be confined to this one somewhat narrow and neglected field. The adjective mathematical is, however, a very necessary qualifier of activity; for during the forty years, from 1839 to 1879, Grassmann had been occupied with other researches. In particular, he had published in the early seventies, an exhaustive dictionary to the Rig-Veda, a work more laborious and in its field no less a standard, even to this day, than the two Ausdehnungslehren. To have accomplished such results in two so different fields is a rarity even in a genius.

Of the twenty-four memoirs in the first part of the volume under review, by far the greater number treat the geometric theories of plane cubics and quartics. It is here that one finds the well known methods developed by Grassmann for the construction of plane curves of degree three and four. Of the other contributions perhaps only two deserve individual notice: those entitled "Sur les différents genres de multiplication" and "Stücke aus dem Lehrbuche der Arithmetik." Grassmann's general definition of multiplication is worth quoting here. He says: To multiply extensive magnitudes, first multiply each by each and in the order indicated the units of which the magnitudes are composed; then multiply these products by the products of their respective coefficients and add. He then goes on to state that other products are merely relative. That was in 1859. The standpoint here outlined was that adopted later by Gibbs. In most of Grassmann's work, whether earlier or subsequent, a more specialized position is maintained relative to products. In no other way can one account for the comparatively meager attention given by Grassmann to the dyadics or polyadics (Lückenausdrücke). The selections from a treatise on arithmetic are in reality nothing but a discussion of the theory of numbers taken broadly. It resembles to a great extent the present method, especially Hilbert's method, of laying down axioms or postulates and building up the analysis therefrom. In form the presentation differs considerably from that now adopted. The words postulate and axiom nowhere occur. In their place stands "Erklärung." In the prefatory remarks the author states that certain parts of the treatment are intended for successive grades in the gymnasium. It is difficult to see how students so immature could profit greatly by the study of such material.

The first memoir on mechanics is again a "Programm" and is intended as an introduction to the subject for students in the gymnasium. Here once more it is difficult to see how the students could have derived much profit from the study. To be sure the presentation is logical and explicit. That seems to be the most serious objection to it. The student would get too much mathematics and logic, too little physics and mechanical intuition from his studies. This would be an excellent argument for having mechanics taught by physicists and engineers rather than by mathematicians, were it not a still better argument for having mathematicians more or less forget their mathematics and really learn mechanics as such before trying to teach it. These two treatises on arithmetic and mechanics are very interesting and highly worth studying. In their way, they are quite distinctive. Grassmann's other work on mechanics is largely a presentation of the subject from the standpoint of the Ausdehnungslehre. The contributions to physics are partly original, partly explanatory. They would not greatly interest anybody now actively engaged in research or in teaching in physical science. This is largely true of all but the highest work done on physics thirty or forty years ago. Mediocrity is short lived in physical science; and it is evident that as a contributor to physics and mechanics Grassmann was not of high rank. This is no reproach: the Ausdehnungslehre and the dictionary to the Rig-Veda are there to silence all specious criticism.

E. B. WILSON.

Die Grundlagen der Bewegungslehre, von einem modernen Standpunkte aus. Von DR. G. JAUMANN, Professor der Physik an der Deutschen Technischen Hochschule in Brünn. Mit 124 Abildungen. Leipzig, Johann Ambrosius Barth, 1905. vi + 421 pp. M. 11; M. 12 gebunden.

IN the author's view the transformation which Fresnel wrought in the theory of light, and Faraday in electromagnetic phenomena, furnishes a prototype for all domains of theoretical physics, mechanics not excepted. The latter, although preserving its old form unaltered, proceeds now in part from points of view which lay quite beyond the reach of its founder and, in this respect, it appears to be nearing a decisive change. This rather remote aim gave direction to the treatment of this book.

It is held that the classical method of deducing a science from the minimum number of experiments and a few fundamental propositions is no longer best; if, indeed, it is admissible at all nowadays. That method furnishes deductive certainty and breadth of development, but it closed the way to progress. The somewhat rigid, dogmatic form of present-day mechanics is due to the manner of its founding. Newton based it upon the very special domain of the motion of discrete rigid bodies in the air. Though this procedure is historically explicable, it can hardly endure indefinitely.

The form of the treatment chosen is that of a systematic text-book of the foundations of mechanics. The pedagogical purpose, of course only a secondary one, has been pursued consistently and seriously. In every appropriate place, the attempt is to make the treatment as easily comprehensible as possible. But where rigor and clearness in main questions called for it, some higher demands have been made upon the reader.

The same point of view is taken regarding the analytic means employed. Advance in theoretical physics depends materially upon the development of suitable mathematical notions. The question of mechanics is always that of disclosing a law to bind observations together. An empirical law for a special series of observations and a general physical law proceed from essentially similar inductions. To discover the form of an empirical law requires choice from amongst appropriate mathematical forms. Without this only naive theories can arise and many geometric relationships must be regarded as erroneous for the fundamental facts of physics.

Reasoning like the foregoing brings the author to the conclusion that it is impossible to desist from employing vector analysis. To avoid narrowing his circle of readers, the author distributes through the book in appropriate places what, in its totality, makes up a brief exposition of the whole theory of vector analysis. The citations for fuller elucidation of the theory are to Gibb's Vector Analysis, edited by E. B. Wilson, New York, 1904, whence the greater part of the notation is taken.

A brief and suggestive introduction sketches lucidly the psychological origin of the mental bias of man which leads him to over-emphasis of the relative importance of mechanical facts and laws as compared with other domains of natural law. As an illustration, the circumstance is cited that on account of the one-sided development of our senses, we unconsciously ascribe greater significance to the change involved in raising a mass of water 424 meters high, or throwing it with an initial velocity of 91 meters per second, than we do to the change involved in heating the mass of water enough to raise its temperature by 1° centigrade, although these effects have been proved to be objectively equivalent.

The compact language and modes of thought of vector analysis make possible a very condensed treatment of a very extended field of phenomena. The topics that are covered with considerable fulness of detail are the following: motion of rigid bodies in the air, dynamic reaction, rigid media, acoustics (embracing pure acoustics and vibroscopy), motion of deformable media, ideal fluids, elastic and viscous media, and the doctrine of force, including forces in space, and forces on surfaces.

For those who have not accustomed themselves to the form of thought of vector analysis, the book will offer some difficulties, despite the author's attempt to simplify matters, but the elegance of the new mode of scientific thought will repay the effort needed to overcome the difficulties, which are not great. The book is quite as interesting and informing in its interpretations of the various results of the operations of vector analysis, as for its value as a higher presentation of the theory of motion.

The author has made a worthy and a more than fairly successful attempt to do a laudable scientific service. He seeks to bring into organic union the most powerful mathematical language and form of thought yet devised, and a broad field of scientific ideas demanding precisely this sort of language for its adequate expression. The undertaking is too well carried out to merit anything but commendation. To complain of a few typographical errors would only augment the volume of hysteria for criticism already too rampant in certain quarters among us.

G. W. MYERS.

Text-Book on the Strength of Materials. By S. E. SLOCUM, B.E., Ph.D., Professor of Applied Mathematics in the University of Cincinnati, and E. L. HANCOCK, M. S., Assistant Professor of Applied Mechanics in Purdue University. Ginn and Co., 1906. xii + 314 pp.

THIS is one of the series of mathematical texts that are being issued under the editorial supervision of Professor Percey

F. Smith, of the Sheffield Scientific School. Both as to content and form this new addition is up to the high standard of excellence for which the series has already earned a reputation.

The authors have aimed to make a text-book on the strength of materials, which should represent the best theory and practice and at the same time be elementary enough for junior classes in technical and engineering schools. To this end they have devoted the first part of the book to theoretical discussions for class use, while the second part is a description of the physical properties of materials, with methods of testing, etc., the latter being intended for use as a sort of laboratory manual. To arouse interest in the subject, practical applications are introduced in Part I, for computation; and in Part II, for observation.

The following phases of the subject are stressed: the definition of the moment of inertia as the shape factor; the graphical method of calculating moments of inertia and centers of gravity; the application of the principle of least work; comparison of column formulas, with graphical illustrations of their relation; precise formulas for torsion of shafts of various cross-sections; simple and correct methods for the strength of hooks, links, springs, etc.; simple formulas for the strength of flat plates; elementary discussion of arches, of retaining walls and foundations; and a separate chapter in Part II on the modern use of reënforced concrete.

The question of most serious import to the writer is whether, on the whole, it is better to divide the student's time, which is now usually given to strength of materials alone, between that science and what is commonly taught in a course by itself under the heading Materials of Construction. Briefly, the book seems trying to do too much. Those who, like the writer, have taught strength of materials to junior engineering students have not found the allotted time any too great for that subject alone. As a student, the writer well recalls that the same truth holds for materials of construction. Even with a full course on the latter subject, but little more than a mere beginning can be made. So brief a treatment as is necessitated by making it a subordinate part of a course in strength of materials would hardly furnish a smattering. Both of these subjects are of fundamental importance to prospective engineers, and neither ought to be curtailed to the extent of giving them both in a single course, unless that course cover an entire year. The practicability of this extension of time is against it. Furthermore, the book does not go into the subject far enough to justify this expenditure of time on the combined course. In the reviewer's opinion, the book has fallen into the inevitable error of too great condensation, viz., superficiality.

The correlation of these two subjects treated is, of course, important. It is also readily admitted that differentiation of subject matters is a necessary part of any true correlation. But the treatment of two subjects separately, one after the other, is not correlating them at all. It is only getting ready for a sound correlation. A real correlation of the subjects would mean furnishing the observational and experimental backing to the theory antecedently, and in close connection with it; and not subsequently and in no particularly close relation to the theory. From the viewpoint of teachability the writer is disposed to question the separation of the subjects, the order of them after they are separated, and the limited scope of both parts of the work.

The work that is given is exquisitely well done. The demonstrations are simple and graphic. It can hardly be maintained that there is either too much or too little illustrating and exemplifying. In these regards the book is unusually well-balanced. Mathematics is used, and freely used, when it helps. So far as the book goes it is mathematical enough to justify the claim to being truly, not naïvely, scientific. Every teacher would find great assistance from it, and many schools will find it well adapted to their purposes and programs.

As a mechanical piece of book-making the text is nothing short of admirable. The cuts — many of them very difficult — the typography, and the arrangement of matter on the page are up to the high standard of the contents of the book.

G. W. MYERS.

NOTES.

THE July number (volume 8, number 3) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "General theory of approximation by functions involving a given number of arbitrary parameters," by J. W. YOUNG; "On derivatives over assemblages," by E. R. HEDRICK; "Geometrie proiettive di congruenza e geometrie proiettive finite," by B. LEVI; "Collineations in a finite projective geometry," by O. VEBLEN; "Geometry in which the sum of the angles of every triangle is two right angles," by R. L. MOORE; "Non-desarguesian and non-pascalian geometries," by O. VEBLEN and J. H. MACLAGAN-WEDDERBURN; "Modular theory of group matrices," by L. E. DICKSON; "Existence proof for a field of extremals tangent to a given curve," by O. BOLZA; "A new form of the simplest problem of the calculus of variations," by G. A. BLISS; "On certain isothermic surfaces," by A. E. YOUNG.

THE University of Illinois has been added to the list of institutions contributing to the support of the *Transactions*.

THE concluding (July) number of volume 8 of the *Annals of Mathematics* contains: "Multiply perfect numbers of four different primes," by R. D. CARMICHAEL; "On a system of parastroids," by R. P. STEPHENS; "A peculiar example in minima of surfaces," by E. R. HEDRICK: "On maximum and minimum values of the modulus of a polynomial," by D. N. LEHMER; "On the minimum surface of revolution in the case of one variable end point," by Miss M. E. SINCLAIR; "On the polynomial convergents of a power series," by M. B. PORTER.

THE annual list of American doctorates published in *Science* presents for the academic year 1906–1907 327 names, of which 168 are credited to the sciences. The following 13 successful candidates offered mathematics as major subject (the titles of the theses are appended): Miss F. A. ALLEN, Wisconsin, "On the determination of cyclic involutions of order three"; C. S. ATCHISON, Johns Hopkins, "Curves with a directrix"; G. D. BIRKHOFF, Chicago, "Asymptotic properties of certain ordinary differential equations with applications to boundary value and expansion problems"; W. C. BRENKE, Harvard, "A contribution to the theory of trigonometric and zonal harmonic series"; H. H. CONOVER, Yale, "On certain problems in the calculus of variations"; A. E. LANDRY, Johns Hopkins, "A geometric interpretation of binary syzygies"; R. MORRIS, Cornell, "On the automorphic functions of the group (0, 3; l_1, l_2, l_3)"; F. W. REED, Virginia, "Singular points in the approximate development of the perturbation function"; A. RANUM, Chicago, "A new kind of congruence groups"; F. R. SHARPE, Cornell, "The general circulation of the atmosphere"; W. B. STONE, Virginia, "The groups of two, three, and four parameters in space and their differential invariants"; A. L. UNDERHILL, Chicago, "Invariants under point transformations in the calculus of variations"; B. M. WALKER, Chicago, "On the resolution of higher singularities of algebraic curves into ordinary double points."

The number of American doctorates in mathematics for each year of the last decade is 11, 13, 11, 18, 8, 7, 14, 21, 11, 13.

THE last annual list of members of the London mathematical society contains 280 names, including 27 Americans and 13 honorary members. The society was founded in 1865, as the outgrowth of the students' mathematical clubs, to a large extent through the efforts of Professor A. DeMorgan, who was made its first president. The last of the original charter members was the late Dr. E. J. Routh. Steps were at once taken to publish papers read before the society, the first volume of the *Proceedings* appearing in 1870, and a volume has appeared every year since. An index of authors of papers published in the first thirty volumes was issued in 1900; in 1902 the size of the page was increased, and larger type employed. In this new series the reports of the meetings appear at the beginning of the volume. The list of papers read at each monthly meeting has regularly appeared in the BULLETIN. In honor of its first president, the society in 1884 founded the DeMorgan medal, which is awarded every three years for exceptional merit. The list of medallists is Professors Cayley, Sylvester, Lord Rayleigh, Klein, S. Roberts, W. Burnside, Greenhill, Baker. The next award will be made in 1908. The officers for the present year are: president, Professor W. BURNSIDE; secretaries, Professor A. E. H. LOVE and Mr. J. H. GRACE; two vice-presidents, treasurer and ten other members of the council. The next election will he held November 14, 1907.

THE *Revista trimestral de Matemáticas*, founded six years ago by Professor J. RUIS Y CASAS of Zaragoza, was discontinued with the completion of number 21, which appeared last spring. In its place a new periodical has been founded, with the title, *Anales de la Facultad de Ciencias de Zaragoza*, which will contain memoirs from the sciences of mathematics, physics, chemistry, astronomy, meteorology and the natural sciences.

THE academy of sciences of Copenhagen announces the following prize problem, for the solution of which it will award the gold medal of the society: "To complete, by new results, the theory of the plane or of an algebraic surface, the points of which correspond reciprocally." Competing memoirs must be written in Danish, Swedish, English, German, French or Latin, and sent to the secretary, Professor H. G. Zeuthen, before October 31, 1908.

THE various German universities offer the following courses in mathematics during the winter semester of 1907–08.

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Analytic geometry, four hours; Applications of elliptic functions, four hours; Examples in conformal mapping, two hours; Colloquium, two hours; Seminar, two hours. — By Professor G. FROBENIUS: Theory of numbers, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: Theory of curves and surfaces, four hours; Linear differential equations and automorphic functions, four hours; Seminar, two hours. — By Professor G. HETTNER: Infinite series, products and continued products, two hours. — By Professor J. KNOBLAUCH: Differential calculus, four hours; Theory of elliptic functions, four hours. — By Professor E. LANDAU: Integral equations, four hours. — By Dr. I. SCHUR: Theory of algebraic equations, four hours; Linear substitutions, two hours. — By Professor R. LEHMANN-FILHÈS: Integral calculus, four hours. — By Dr. J. ASCHKINASS: Elements of higher mathematics for students of science, four hours.

UNIVERSITY OF BONN. — By Professor E. STUDY: Introduction to the theory of functions, four hours; Introduction to quaternions, one hour; Seminar, two hours. — By Professor F. LONDON: Advanced calculus, four hours; with exercises, one hour; Synthetic geometry, two hours; Exercises in descriptive geometry, two hours; Seminar, two hours. — By Professor G.

KOWALEWSKI: Analytic geometry of the plane and of space, four hours; with exercises, one hour; Introduction to the theory of groups of transformations, two hours.— By Dr. E. SCHMIDT: Partial differential equations, four hours.

UNIVERSITY OF GÖTTINGEN.— By Professor F. KLEIN: Elementary mathematics from a higher standpoint, four hours; Seminar, two hours.— By Professor D. HILBERT: Theory of partial differential equations, four hours; Integral equations, two hours; Seminar, two hours.— By Professor H. MINKOWSKI: Theory of functions, four hours; Selected chapters of the theory of numbers, two hours; Seminar, two hours.— By Professor C. RUNGE: Graphical methods, four hours; with exercises, two hours; Seminar, two hours.— By ——: The basis of insurance, three hours; Seminar.— By Professor E. ZERMELO: The mathematical basis of logic, two hours. — By Dr. G. HERGLOTZ: Algebra, four hours; Exercises in differential equations, two hours.— By Dr. C. CARATHÉODORY: Advanced calculus, four hours; with exercises, two hours.— By Dr. P. KOEBE: Quadric surfaces, two hours.— By Professor W. VOIGT: Mechanics, four hours.— By Dr. M. ABRAHAM: Theory of elasticity, two hours.

UNIVERSITY OF HALLE. — By Professor G. CANTOR: Theory of elliptic functions, four hours; Selected chapters of analytic mechanics, two hours; Seminar, two hours — By Professor A. WANGERIN: Theory of potential and spherical harmonics, four hours; Integral calculus with exercises, four hours; Seminar, two hours. — By Professor A. GUTZMER: Calculus of variations, four hours; Analytic geometry of space, four hours; Axonometry and perspective, two hours. — By Professor V. EBERHARD: Numerical equations and iterated functions, two hours; Theory of numbers, II, two hours; Colloquium, two hours. — By Dr. F. BERNSTEIN: Theory and application of definite integrals, four hours; Mathematics of insurance, one hour.

UNIVERSITY OF JENA. — By Professor F. HAUSSNER: Integral calculus with exercises, five hours; Algebra, four hours; Analytic geometry of space, four hours; Seminar, two hours; Proseminar, two hours. — By Professor J. THOMAE: Differential equations, four hours; Seminar, two hours. — By Professor G. FREGE: Analytic mechanics, four hours; Symbolic methods, one hour. — By Professor R. RAU: Graphical methods in statics and mechanics, three hours.

UNIVERSITY OF LEIPZIG. — By Professor C. NEUMANN: Analytic mechanics, four hours.— By Professor A. MAYER: Calculus of variations, four hours. — By Professor O. HÖLDER: Elliptic functions, four hours; Seminar, two hours. — By Professor K. ROHN: Analytic geometry of space, four hours; Descriptive geometry, two hours; with exercises, two hours. — By Professor F. HAUSDORFF: Calculus, four hours; with exercises, two hours. — By Professor H. LIEBMANN: Theory and application of determinants, two hours; Non-euclidean geometry, two hours.

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN: Differential calculus, five hours; Theory of abelian functions, four hours; Seminar, one and a half hours. — By Professor A. VOSS: Plane analytic geometry, four hours; Analytic mechanics, I, four hours; Seminar, two hours. — By Professor A. PRINGSHEIM: Theory of infinite series and of similar limiting processes, four hours; Elements of the theory of functions, five hours. — By Professor K. DOEHLEMANN: Descriptive geometry, I, five hours; with exercises, three hours; Synthetic geometry, with exercises, five hours; Lines in art, one hour. — By Professor H. BRUNN; Theory of aggregates, four hours. — By Dr. K. HARTOGS: Integral calculus, six hours. — By Dr. O. PERRON: Elementary geometry and trigonometry, three hours.

UNIVERSITY OF STRASSBURG. — By Professor T. REYE: New methods in the analytic geometry of space, four hours; Mathematical theory of elasticity of rigid bodies, two hours; Seminar, two hours. — By Professor H. WEBER: Calculus, four hours; Differential equations of mathematical physics, two hours; Seminar, hours. — By Professor M. SIMON; Fundamental conceptions of mathematics and mechanics, four hours. — By Professor J. WELLSTEIN: Selected chapters of the theory of functions, four hours; Determinants and matrices, two hours; Seminar, two hours. — By Professor H. E. TIMERDING: Plane analytic geometry, with exercises, four hours; Graphical statics with exercises, two hours; Vector analysis, two hours; Seminar, two hours. — By Dr. S. EPSTEIN: Introduction to higher mathematics for students of natural science, four hours; Seminar, two hours.

AT the University of Göttingen Professor L. PRANDTL has been promoted to a full professorship of mathematics; Professor

M. BRENDEL has been appointed professor of mathematics at the commercial academy of Frankfurt am Main; Dr. O. TOEPLITZ has been appointed docent in mathematics.

PROFESSOR J. DRACH, of the University of Poitiers, has been transferred to a professorship of analysis at the same institution.

DR. M. GROSSMANN, of the University of Basel, has been appointed professor of descriptive geometry at the technical of Zürich, as successor to Professor W. Fiedler, who has retired from active service.

DR. F. HAUSDORFF has been elected an associate member of the academy of sciences at Leipzig.

PROFESSOR J. HORN, of the mining academy at Clausthal, has been appointed professor of mathematics at the technical school of Darmstadt.

DR. L. SCHRUTKA, has been appointed docent in mathematics at the University of Vienna.

PROFESSOR G. HESSENBERG, of the technical school at Charlottenburg, has been appointed professer of mathematics at the agricultural institute at Bonn-Poppelsdorf.

PROFESSOR A. KORN, of the University of Munich, has been elected a member of the academy of sciences at Halle.

DR. M. KUTTA has been promoted to an associate professorship of mathemaics at the technical school of Munich.

PROFESSOR R. D. v. STERNECK, of the University of Czernowitz, has been appointed professor of mathematics at the University of Graz.

DR. R. WEBER, of the University of Heidelberg, has been appointed associate professor of mathematics at the University of Rostock.

DR. G. HERGLOTZ, of the University of Göttingen, has been appointed associate professor of mathematics at the University of Freiburg, Switzerland.

PROFESSOR H. v. MANGOLDT, of the technical school at Danzig, has been decorated with the order of the red eagle of the third class. Professor M. DISTELI, of the technical school at Dresden, has received the title of knight of the cross, of the first class.

THE following promotions to full professorships of mathematics have been made in the French provincial universities: Professor J. CLAIRIN, at the University of Lille; Professor R. LE VAVASSEUR, at the University of Lyons; Professor A. PARAF, at the University of Toulouse.

MR. J. MERCER has been appointed assistant lecturer in mathematics at the University of Liverpool.

AT the University of Illinois the following changes have been made in the mathematical staff: Professor G. A. MILLER has been promoted to a full professorship; Professor E. J. WILCZYNSKI, formerly at the University of California, has been appointed associate professor; Dr. C. H. SISAM was advanced from the grade of instructor to associate; Mr. B. M. MATHEWS has been appointed assistant. As at present constituted the department consists of three professors, one associate professor, three assistant professors, one associate, five instructors, and three assistants.

THE following changes are announced at the University of Missouri: Dr. W. D. A. WESTFALL has been appointed assistant professor of mathematics and granted leave of absence for one year to study in Europe; Dr. OTTO DUNKEL has been appointed instructor; Dr. L. INGOLD returns from his leave of absence; Mr. L. L. SILVERMAN has been appointed acting instructor; Miss A. PAYNE and Miss A. M. LIEPSNER have been appointed assistants; Miss M. S. WALKER has been appointed instructor and granted leave of absence to continue her studies.

PROFESSOR T. F. HOLGATE, of Northwestern University, has been granted leave of absence for the present academic year, and will spend most of the time at Cambridge, England. Professor D. R. CURTISS has been promoted to a full professorship of mathematics.

DR. A. B. PIERCE, of the University of Michigan, has been promoted to an assistant professorship of mathematics.

PROFESSOR O. P. AKERS, of Allegheny College, has been made full professor of mathematics.

DR. J. A. EIESLAND has been promoted to the chair of mathematics at the United States Naval Academy, Annapolis.

DR. W. C. BRENKE has been appointed adjunct professor of mathematics at the University of Nebraska.

PROFESSOR G. D. GABLE, of Parsons College (Iowa), has been appointed professor of mathematics at the University of Wooster, Ohio.

DR. W. H. BUSSEY, of Columbia University, has been appointed assistant professor of mathematics at the University of Minnesota.

PROFESSOR A. L. RHOTON, of the Southwestern University of Tennessee, has been appointed professor of mathematics at Georgetown College, Kentucky.

PROFESSOR R. C. ARCHIBALD, of the Mount Allison Ladies' College, has been appointed professor of mathematics at Acadia University, Wolfsville, N. S.

THE following appointments have been made at the University of Pennsylvania: Dr. F. H. SAFFORD has been promoted to an assistant professorship of mathematics; Messrs. M. J. BABB and L. O'SHAUGHNESSY have been named instructors in mathematics.

DR. R. P. STEPHENS, of Wesleyan University, has been appointed adjunct professor of mathematics at the University of Georgia. Mr. H. B. CAMP has been appointed instructor in mathematics at Wesleyan.

AT Syracuse University, Mr. D. PRATT has been appointed assistant professor of mathematics, and Mr. H. F. HART has been appointed instructor in mathematics.

DR. E. SWIFT has been appointed instructor in mathematics at Princeton University.

MR. R. L. BÖRGER, of the University of Florida, has been appointed instructor in mathematics at the University of Illinois.

MR. G. I. GAVETT, of Cornell University, has been appointed instructor in mathematics at the University of Washington.

MR. A. C. KRATHWOHL has been appointed tutor in mathematics at Barnard College, Columbia University.

MR. C. A. TOUSSAINT, of Columbia University, has been appointed tutor in mathematics at the College of the City of New York.

MR. W. V. LOVITT has been appointed instructor in mathematics at the University of Washington, Seattle, Washington.

MR. W. T. AUDE, of Colgate University, has been appointed instructor in mathematics at the Carnegie Institute in Pittsburgh.

MR. C. H. FORSYTH has been made assistant in mathematics at the University of Illinois.

MISS M. E. SINCLAIR, of the University of Nebraska, has been appointed instructor in mathematics at Oberlin College.

PROFESSOR E. C. COLPITTS, of the Georgia School of Technology, has been appointed professor of mathematics at Emporia College, Kansas.

DR. OREN ROOT, for 27 years professor of mathematics at Hamilton College, died August 26, at the age of 69 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BROGGI (U.). Traité des assurances sur la vie avec développements sur le calcul des probabilités. Paris, Hermann, 1907. 8vo. 11+309 pp.

DINI (U.). Lezioni di analisi infinitesimale. Vol. I. Calcolo differenziale. Pisa, Nistri, 1907. 8vo. 720 pp. L. 22.00

DOLGE (P.). Ueber Bernoullische Zahlen und Funktionen, welche zu einer Fundamentaldiskriminante gehören, und deren Anwendung auf die Summation unendlicher Reihen. (Progr.) Dresden, 1907. 8vo. 44 pp.

EGERER (H.). Ueber die Kurve der Ecken der Vierseite, die von den gemeinsamen Tangenten eines festen Kegelschnitts und der Kegelschnitte eines Büschels gebildet werden. (Diss.) Erlangen, 1906. 8vo. 53 pp.

FACK (M.). Zur didaktischen Darstellung von Stoffen aus der niedern und höheren Mathematik. (Progr.) Weimar, 1907. 8vo. 53 pp.

FAZZARI (G.). Breve storia della matematica dai tempi antichi al medio evo. Palermo, Sandron, 1907. 16mo. 267 pp. L. 4.00

FOETHKE (E.). Anwendung des erweiterten Euklidischen Algorithmus auf Resultantenbildungen. (Diss.) Königsberg, 1907. 8vo. 75 pp.

GILLESPIE (D. C.). Anwendungen des Unabhängigkeitssatzes auf die Lösung der Differentialgleichungen der Variationsrechnung. (Diss.) Göttingen, 1906. 8vo. 75 pp.

GRABER. Zur Zylinder- und Kugelberechnung. (Progr.) Kottbus, 1907. 8vo. 39 pp.

GUTSCHE (O.). Geometrische Untersuchungen über inverse Kurven von Kegelschnitten. (Progr.) Breslau, 1907. 8vo. 44 pp.

KRUSCHE (A.). Ueber Kurven und Flächen, welche sich aus geradlinigen Flächen 2 ten Grades durch gemeinsame Lote zwischen den Erzeugenden ableiten lassen. (Diss.) Breslau, 1905. 8vo. 58 pp.

LORIA (G.). Vorlesungen über darstellende Geometrie. Autorisierte, nach dem italienischen Manuskript bearbeitete deutsche Ausgabe von F. Schütte. Teil I: Die Darstellungsmethoden. Leipzig, Teubner, 1907. 8vo. 11 + 219 pp. Cloth. (B. G. Teubner's Sammlung von Lehrbüchern, Vol. XXV.) M. 6.80

MEISSNER (W.). Zur Theorie des Strahlungsdruckes. (Diss.) Berlin, 1907. 8vo. 75 pp.

OPPLIGER (A.). Historisch-kritische Untersuchung über die Theorie der Kugelfunktionen. (Diss.) Bern, 1906. 8vo. 62 pp.

PEREZ DE MUNOZ (F.). Introducción al estudio del cálculo de Cuaterniones y otras algebras especiales. Madrid, Garcia, 1907. 8vo.

PETERS (J. M.). Die Summen der ganzen positiven Potenzen der natürlichen Zahlen und ihre Darstellung als Funktionen der Summe dieser Zahlen. (Progr.) Cöln, 1907. 4to. 21 pp.

PYRKOSCH. Das Büschel von Kurven 4ter Ordnung mit 3 festen Doppelpunkten und 4 festen einfachen Punkten. (Progr.) Breslau, 1907. 8vo. 16 pp.

RICHERT (P.). Die ganzen rationalen Funktionen der ersten drei Grade und ihre Kurven. Exponentialreihen höherer Grade. (Progr.) Berlin, Weidmann, 1907. 8vo. 77 pp. M. 1.00

WHITEHEAD (A. N.). The axioms of descriptive geometry. Cambridge, University Press, 1907. 8vo. 82 pp. 2s. 6d.

WIRTH (H.). Beiträge zur Theorie der Abbildungen durch reciproke radii vectores. (Progr.) Wolgast, 1907. 8vo. 32 pp.

ZEHME. Eigenschaften des Krümmungsschwerpunktes ebener Kurven. Teil II. (Progr.) Arnstadt, 1907. 4to. 24 pp.

II. ELEMENTARY MATHEMATICS.

ARNOUX (V.). Deux mille cinq cents problèmes d'arithmétique et de géométrie pratique. (Nombres décimaux, fractions, règles de trois simple et composée, intérêts simple et composés, rentes sur l'état, etc.). Problèmes d'examen. 10e édition, revue et augmentée. Paris, Larousse, 1907. 12mo. 288 pp. F. 2.00

BAKER (W. M.). A key to algebraic geometry. London, Bell, 1907. 8vo. 7s. 6d.

BARNARD (S.) and CHILD (J. M.). A new geometry for middle forms, being parts 1, 2 and 3 of "A New Geometry" and equivalent to Euclid, Books 1–4, together with additional matter. London, Macmillan, 1907. 12mo. 438 pp. 3s. 6d.

BEARD (W. S.). Easy exercises in algebra for beginners. London, Methuen, 1907. 12mo. 118 pp. 1s. 6d.

BOURLET (C.). Eléments d'algèbre, rédigés conformément aux programmes du 31 mai 1902 et aux programmes de l'enseignement primaire supérieur. 2e édition. Paris, Hachette, 1907. 16mo. 388 pp. F. 3.00

BROOKSMITH (E. J.). Mathematical papers. For admission into the Royal military academy and the Royal military college, for the years 1897–1906. London, Macmillan, 1907. 12mo. 6s.

CHILD (J. M.). See BARNARD (S.).

DEAKIN (R.). Key to the new matriculation algebra. London, Clive, 1907. 12 mo. 342 pp. 5s. 6d.

——. New geometry papers. London, Clive, 1907. 12mo. 1s.

ERNEWELL (E. R.). Regeln der Mathematik. I. Lehrgang. Die Geometrie der Ebene (Planimetrie) und die Grundlehren der Stereometrie (Raumlehre) für den Schul- und Selbstunterricht. Nach völlig neuer, streng wissenschaftlicher Methode für jedermann verständlich dargestellt. Berlin, Selbstverlag, 1907. 8vo. 4 + 100 pp. M. 10.00

GREVY (A.). Géométrie dans l'espace à l'usage des élèves des classes de première C et D (Programme du 27 juillet, 1905). 2e édition. Paris, Vuibert, 1907. 8vo. 214 pp. F. 2.00

JACKSON (W. H.). Elementary solid geometry, including the mensuration of the simple solids. New York, Longmans, 1907. 12mo. 12+159 pp. $0.65

KELLER (S. M.). Mathematics for engineering students: algebra and trigonometry. New York, Van Nostrand, 1907. 8vo. 4 + 250 pp. (Carnegie Technical Schools text-books). Cloth. $1.50

KÖHLER (A.). See LIEBER (H.).

KUTNEWSKY (M.). See MÜLLER (H.).

LAZZERI (G.) e PESCHI (G.). Complementi d'algebra per l'ammissione alla r. accademia navale. 2e edizione. Livorno, Unione poligrafica livornese, 1907. 8vo. 120 pp. L. 1.50

LEBON (E.). Géométrie cotée et géométrie descriptive (conforme aux programmes du 27 juillet 1905 pour l'enseignement secondaire). 2e édition. Paris, Delalain, 1907. 8vo. 198 pp.

——. Sur la construction d'une table de caractéristiques relatives à la base 30030 des facteurs premiers d'un nombre inférieur à 901800900. Paris, Imprimerie nationale, 1907. 8vo. 7 pp.

LIEBER (H.) und KÖHLER (A.). Arithmetische Aufgaben. 4te Auflage. Berlin, Simion, 1907. 8vo. 8 + 232 pp. M. 2.70

MARKS (C. I.). Mathematical questions and solutions. New Series, Vol. II. London, Hodgson, 1907. 8vo. 6s. 6d.

MICHALITSCHKE (A.). Die mathematische Geographie und die Himmelskunde im elementaren Unterrichte an der Volks- und an der Mittelschule. Ein Beitrag zur Methodik des Gegenstandes. Wien, Deuticke, 1907. 8vo. 3 + 24 pp. M. 0.80

MÜLLER (H.). Sammlung von Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie. Ausgabe B, für reale Anstalten und Reformschulen. Teil II: 2te Auflage. Leipzig, Teubner, 1907. 8vo. 11+304 pp. Cloth. M. 3.00

PESCHI (G.). See LAZZERI (G.).

RICHTER (O.). Dreistellige logarithmische und trigonometrische Tafeln. Leipzig, Teubner, 1907. 8vo. 10 pp. M. 0.20

SCHULZ (E.). Leitfaden der Planimetrie für gewerbliche Lehranstalten. Teil II. 4te Auflage. Essen, Baedeker, 1907. 8vo. 4+98 pp. Boards M. 1.00

TIMPENFELD (P.). Tabellen der Quadrate von 1 bis 10,000, Kuben von 1 bis 2,500, Quadrat- und Kubikwurzeln von 1 bis 1,000, Kreisumfänge und—inhalte von 1 bis 1,000. 4te Auflage. Dortmund, Krüger, 1907. 8vo. 109 pp. Cloth. M. 3.50

TREUTLEIN (P.). Mathematische Aufgaben aus den Reifeprüfungen der badischen Mittelschulen. Teil I: Aufgaben. Leipzig, Teubner, 1907. 8vo. 10 + 158 pp. Cloth. M. 2.80

VOLK (K. G.). Die Elemente der neueren Geometrie unter besonderer Berücksichtigung des geometrischen Bewegungsprinzips. Für die oberen Klassen höherer Lehranstalten und zum Selbststudium bearbeitet. Leipzig, Teubner, 1907. 8vo. 8 + 77 pp. Boards. M. 2.00

WRIGHT (J.) Exercises in concrete geometry. London, Heath, 1907. 12mo. 2s.

III. APPLIED MATHEMATICS.

ARCHER (J. A.). Compound interest, annuity and sinking fund tables. London, Shaw, 1907. 8vo. 180 pp. 8s. 6d.

BALDWIN (S.). Grundlagen der Mechanik. Kursus der Nikolajewschen Ingenieurschule. (Russian.) St. Petersburg, 1907. 8vo. M. 10.00

BOERNER (F.). Statische Tabellen, Belastungsangaben und Formeln zur Aufstellung von Berechnungen für Baukonstruktionen. 2te erweiterte Auflage. Berlin, Ernst, 1907. 8vo. 8 + 196 pp. Boards. M. 3.50

BRIGGS (W.) and BRYAN (G. H.). Intermediate hydrostatics, including matriculation hydrostatics. London, Clive, 1907. 8vo. 272 pp. 3s. 6d.

BRYAN (G. H.). Thermodynamics. An introductory treatise dealing mainly with first principles and their direct applications. New York, Lemcke, 1907. 8vo. 14 + 204 pp. Cloth. $2.35

——. See BRIGGS (W.).

EMDEN (R.). Gaskugeln. Anwendungen der mechanischen Wärmetheorie auf kosmologische und meteorologische Probleme. Leipzig, Teubner, 1907. 8vo. 6 + 498 pp. Cloth. M. 13.00

EYMER (A.). Versuche über die Abhängigkeit der Dielektricitätsconstante von mechanischen und thermischen Einwirkungen. (Diss.) Marburg, 1906. 8vo. 43 pp.

FUCHS (P.). Formeln und Tabellen der Wärmetechnik. Zum Gebrauch bei Versuchen in Dampf-, Gas-, und Hüttenbetrieben. Berlin, Springer, 1907. 8vo. 7 + 43 pp. Cloth. M. 2.00

GÖLDEL (P.). See RAMISCH (G.).

HELMERT (F. R.). Die Ausgleichungsrechnung nach der Methode der kleinsten Quadrate mit Anwendungen auf die Geodäsie, die Physik und die Theorie der Messinstrumente. 2te Auflage. Leipzig, Teubner, 1907. 8vo. 18 + 578 pp. Cloth. M. 16.00

HOELLING (J. H.). Untersuchungen über die Bewegung des Planeten (13) Egeria. (Diss.) Berlin, 1907. 4to. 30 pp.

HOLETSCHEK (J.). Ueber die scheinbare Verlängerung eines Kometenschweifes beim Durchgang der Erde durch die Ebene der Kometenbahn. Wien, 1906. 8vo. 24 pp.

JAMIESON (A.). See MUNRO (J.).

KLUG (R.). Definitive Bahnbestimmung des Kometen 1826. Wien, 1906. 4to. 65 pp.

LE GUILCHER (J. M.). Manuel du mécanicien breveté de la marine de commerce (Arithmétique, algèbre, système métrique, géométrie, mécanique, physique, électricité, résistance des matériaux). Ouvrage rédigé conformément aux programmes des examens. Paris, Challamel, 1907. 8vo. 412 pp.

LUDWIG (W.). Lehrbuch der politischen Arithmetik. Wien, Fromme, 1907. 8vo. 4 + 188 pp. M. 3.75

MUNRO (J.) and JAMIESON (A.). A pocket-book of electrical rules and tables for the use of electricians and engineers. London, Griffin, 1907. 18mo. 8s. 6d.

RAMISCH (G.) und GÖLDEL (P.). Bestimmung der Stärken, Eisenquerschnitte und Gewichte von Eisenbetonplatten. Zahlen-Tafeln für freiaufliegende halb- und ganz eingespannte Platten und beliebig gewählte Spannungswerte für Eisen und Beton, und für Säulen aus Eisenbeton. Berlin, Tonindustrie-Zeitung, 1906. 8vo. 42 pp. Cloth. M. 3.00

SPANGLER (H. W.). Notes on thermodynamics. The derivation of the fundamental principles of thermodynamics and their application to numerical problems. 3d edition. New York, Wiley, 1907. 12mo. 6 + 74 pp. Cloth. $1.00

STEINER (F.). Vermessungskunde. Anleitung zum Feldmessen, Höhenmessen, Lageplan- und Terrainzeichnen. Herausgegeben von Emil Burok. 2te Auflage. Halle, Knapp, 1907. 8vo. 8 + 156 pp. M. 4.80

STOLLE (R.). See WEICHERT (A.).

WEBER (G.). Die Berechnung und Konstruktion der Turbinen und Schützenzüge mit besonderer Berücksichtigung der Francis-Turbine. Leipzig, Schäfer, 1907. 8vo. 131 pp. Boards. M. 8.00

WEICHERT (A.) und STOLLE (R.). Praktisches Maschinenrechnen. Eine Zusammenstellung der wichtigsten Erfahrungswerte aus der allgemeinen und angewandten Mechanik in ihrer Anwendung auf den praktischen Maschinenbau. Erläutert durch zahlreiche für die Praxis verwendbare Beispiele. Berlin, Seydel, 1907. Teil I: Arithmetik und Algebra. Eine leichtfassliche Darstellung der für Maschinenbauer unentbehrlichen Gesetze der Arithmetik und des allgemeinen Buchstabenrechnens. 8 + 120 pp. M. 2.00

ZEUNER (G. A.). Technical thermodynamics, translated from the 5th complete and revised edition of "Grundzüge der mechanischen Wärmetheorie." Authorized translation by J. F. Klein. In 2 vols. Vol. I. New York, Van Nostrand, 1907. 8vo. Cloth. $2.00

THE FOURTEENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

MANY members of the Society recall with pleasure the summer meeting and colloquium held at Ithaca in 1901. After an interval of six years the summer meeting, the fourteenth of the series, was again convened at Cornell University on Thursday and Friday, September 5–6, 1907. By close economy of time the scientific proceedings were condensed into two sessions on Thursday and a morning session on Friday. Friday afternoon was devoted to an excursion on Lake Cayuga, Mr. H. H. Westinghouse, of the University, having kindly placed his steam yacht at the disposal of the members. The evening gatherings at the Town and Gown Club also furnished pleasant opportunities for social intercourse.

The first session opened with an address of welcome by Professor Wait. At the close of the meeting resolutions were adopted expressing the Society's appreciation of the generous hospitality of the University and its officers.

The attendance included the following forty-seven members of the Society:

Professor O. P. Akers, Professor G. A. Bliss, Mr. W. C. Brenke, Professor W. G. Bullard, Dr. W. B. Carver, Professor F. N. Cole, Professor L. L. Conant, Professor E. W. Davis, Professor S. C. Davisson, Dr. E. L. Dodd, Dr. F. J. Dohmen, Professor W. P. Durfee, Professor L. P. Eisenhart, Professor H. B. Fine, Professor W. B. Fite, Professor W. B. Ford, Professor A. S. Gale, Professor C. N. Haskins, Professor E. R. Hedrick, Professor J. I. Hutchinson, Dr. L. C. Karpinski, Professor O. D. Kellogg, Professor T. E. McKinney, Professor James McMahon, Professor W. H. Metzler, Mr. E. A. Miller, Professor G. A. Miller, Dr. R. L. Moore, Mr. F. W. Owens, Dr. Arthur Ranum, Professor E. D. Roe, Professor D. A. Rothrock, Dr. F. R. Sharpe, Miss M. E. Sinclair, Dr. C. H. Sisam, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Professor J. H. Tanner, Professor Anna L. Van Benschoten, Professor E. B. Van Vleck, Professor Oswald Veblen, Professor L. A. Wait, Dr. W. D. A. Westfall, Professor H. S. White, Professor J. M. Willard, Professor Alexander Ziwet.

The President of the Society, Professor H. S. White, occupied the chair, being relieved by Professors Fine and E. B. Van Vleck. The Council announced the election of the following persons to membership in the Society: Mr. Thomas Buck, University of Chicago; Mr. Arnold Dresden, University of Chicago; Mr. T. H. Hildebrandt, University of Chicago; Mr. W. J. King, Harvard University; Mr. J. O. Mahoney, High School, Dallas, Texas; Dr. J. F. Messick, Randolph-Macon College; Mr. H. W. Powell, College of the City of New York. Six applications for membership in the Society were received. The total membership is now 569.

The following papers were read at the summer meeting:

(1) Professor L. E. Dickson: "Modular theory of group matrices."

(2) Professor W. B. Ford: "Sur les équations linéaires aux différences finies."

(3) Professor R. D. Carmichael: "On the classification of plane algebraic curves possessing fourfold symmetry with respect to a point."

(4) Professor R. D. Carmichael: "Note on certain inverse problems in the simplex theory of numbers."

(5) Dr. W. B. Carver: "The ten special $\Gamma^4_{6,2}$ configurations in the Pascal hexagram."

(6) Professor E. O. Lovett: "Generalization of a problem of Bertrand in mechanics."

(7) Professor E. O. Lovett: "The invariants of a group which occurs in the problem of n bodies."

(8) Professor E. R. Hedrick: "A peculiar example in the theory of surfaces."

(9) Professor E. R. Hedrick: "A smooth closed curve composed of rectilinear segments."

(10) Professor R. D. Carmichael: "On certain transcendental functions defined by a symbolic equation."

(11) Dr. D. C. Gillespie: "On the canonical substitution in the Hamilton-Jacobi canonical system of differential equations."

(12) Professor G. A. Miller: "The invariant substitutions under a substitution group."

(13) Professor G. A. Miller: Methods of determining the primitive roots of a number."

(14) Professor Virgil Snyder: "On a special algebraic curve having a net of minimum adjoint curves."

(15) Professor James McMahon: "The differential geometry of the vector field. Second paper: lamellar field."

(16) Professor L. E. Dickson: "Commutative linear groups."

(17) Professor L. E. Dickson: "A simple derivation of the canonical forms of linear transformations."

(18) Professor Edward Kasner: "Geometric interpretation of integrating factors."

(19) Professor Edward Kasner: "The conformal representation of geodesic circles."

(20) Mr. A. R. Schweitzer: "On the relation of right-handedness in geometry."

(21) Dr. F. L. Griffin: "On the law of gravitation in the binary systems, II."

(22) Dr. F. L. Griffin: "Certain trajectories common to different laws of central force."

(23) Professor E. W. Davis: "Colored imaginaries. I, Imaginaries in the plane."

(24) Professor E. W. Davis: "Colored imaginaries. II, Imaginaries in space."

(25) Dr. C. H. Sisam: "On the equations of quartic surfaces in terms of quadratic forms."

(26) Professor Virgil Snyder: "On the range of birational transformation of curves having genus greater than that of the canonical form."

(27) Professor G. A. Miller: "Third report on recent progress in the theory of groups of finite order."

(28) Professor Oswald Veblen: "Continuous increasing functions of ordinal numbers."

(29) Professor H. S. White and Miss K. G. Miller: "Note on Lüroth's type of plane quartic curve."

(30) Professor W. B. Fite: "Concerning the degree of an irreducible linear homogeneous group."

(31) Dr. Arthur Ranum: "Concerning linear substitutions of finite period with rational coefficients."

(32) Professor R. P. Stephens: "Certain curves of class n having $n-2$ tangents in any given direction."

(33) Professor Anna L. Van Benschoten: "Curves of genus 4 which remain invariant under birational transformation."

(34) Miss M. E. Sinclair: "On a discontinuous solution in the problem of the surface of revolution of minimum area."

(35) Dr. Maurice Fréchet: "Sur les opérations linéaires (troisième note)."

(36) Professor A. G. GREENHILL: "The elliptic integral in electromagnetic theory."

Dr. Fréchet's paper was communicated to the Society through Professor E. B. Van Vleck. In the absence of the authors, Dr. Gillespie's paper was presented by Professor Snyder, Dr. Griffin's papers by Professor Veblen, Professor Greenhill's paper by Professor McMahon, and the papers of Professor Dickson, Professor Carmichael, Professor Lovett, Professor Kasner, Mr. Schweitzer, Professor Stephens and Dr. Fréchet were read by title.

Professor Dickson's first paper was published in the July number of the *Transactions*. Professor Miller's first paper appeared in the October BULLETIN. Professor Miller's report, Professor Snyder's first paper, and Professor Carmichael's second paper are published in the present number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

2. Professor Ford's paper considers the linear difference equation

$$a_0(x)\Delta^n y + a_1(x)\Delta^{n-1}y + \cdots + a_n(x)y = 0,$$

and obtains certain results analogous to those established during recent years by Dini for the linear differential equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)y = 0.$$

Four general theorems are established concerning the existence and character of the solutions of the above difference equation, the last three of which are especially adapted to the study of the solutions $y(x)$ in the neighborhood of the point $x = \infty$. The results obtained find noteworthy application in the study of the convergence of algebraic continued fractions, as will appear more at length subsequently. The paper appeared in the last number (volume 13, number 4) of the *Annali di Matematica.*

3. Professor Carmichael considers the classification of plane algebraic curves possessing fourfold symmetry with respect to a point. The discussion is a generalization of his previous paper on quartic curves possessing the same property. It results in the following theorem:

Excluding curves consisting of isolated points only, the only loci of the nth order possessing fourfold symmetry with respect to a point taken as the origin are those whose rectangular cartesian equations are of one of the forms

$$\Sigma[a_u(x^t y^s + (-1)^t x^s y^t)] = 0, \qquad \Sigma[a_u(x^t y^s - (-1)^t x^s y^t)] = 0,$$

where n must be even and $s = 0, 1, 2, \cdots, n$; $t = s, s + 1, \cdots, n$; subject to the condition that $t + s = 0, 2, 4, \cdots, n$. The paper will be published in the *Annals of Mathematics*.

5. In his papers, "Sur quelques théorèmes, etc.," *Crelle's Journal*, volume 31 (1846), and "On Pascal's theorem," *Quarterly Journal*, volume 9 (1868), Cayley called attention to a peculiar special case of the configuration $\Gamma^4_{6,2}$ which occurs ten times in the Pascal hexagram. In a paper "On the Cayley-Veronese configurations," *Transactions*, volume 6 (1905), Dr. Carver showed that there is a pencil of six conics connected with any $\Gamma^4_{6,2}$. The pencils connected thus with the special $\Gamma^4_{6,2}$'s in the Pascal hexagram are studied in the present paper, and their relation to the fundamental conic of the hexagon is shown. The notation used is defined in the author's paper cited above.

6. In the plane xoy let a point (x, y) be acted upon by an acceleration force whose components are certain functions of the coordinates of the point. Professor Lovett proposes to examine in what cases all the trajectories, whatever be the initial conditions, are curves of the family defined by the differential equation

$$C(\alpha, \beta, \gamma, \delta, \epsilon) \equiv \alpha(12)^2(15) + \beta(12)^2(24) + \gamma(12)(13)(14) + \delta(13)^3 + \epsilon(12)(13)(23) = 0,$$

where α, β, γ, δ, ϵ are constants, and (i, j) is written for $x^{(i)}y^{(j)} - x^{(j)}y^{(i)}$, the indices indicating derivations with regard to any independent variable. If the left member is $C(9, 45, -45, 40, -90)$, we have the form in which Stéphanos has written the differential equation of the conics first given by Monge.

This problem was studied by Bertrand* in the case where the force has in general a unique direction at every point and the trajectories are conic sections. Bertrand showed that in this case the force ought either to pass through a fixed point or be parallel to a fixed direction. Stéphanos* has recently examined the general case of conic trajectories when the force has not necessarily a unique direction; he finds that the problem admits of no other solution than those included in the case considered by Bertrand, and shows that every force under which a point describes a conic section, whatever be the initial conditions, always passes through a fixed point or remains parallel to a fixed direction. Finally by applying his method to the determination of the explicit forms of the forces in the case of conics Stephanos rediscovers the known theorems of Darboux and Halphen.†

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*Bertrand: "Note sur un problème de la mécanique," *Comptes rendus*, vol. 84, p. 731. Bertrand had formulated the problem in a previous note "Sur la possibilité de deduire d'une seule les lois de Kepler et le principe de l'attraction," ibid., p. 673.

In the present note free use is made of the method and formulas developed by Stéphanos in his memoir to which reference has been made. The analytic conditions determining the forces are set up and transformed with a view to make appear the condition for central forces or forces parallel to a fixed direction; it results that this condition may be satisfied and one of the parameters in the above differential equation remain absolutely arbitrary, that is, if α, γ, δ are in the ratios $9 : -45 : 40$, and ϵ equals -2β, the forces under which a point describes the corresponding curve, whatever be the initial conditions, either pass through a fixed point or remain parallel to a fixed direction. Furthermore in this case, which obviously includes the conic sections, the differential equations to be integrated for the forces are identical in form to those encountered by Stéphanos; his results and those of Darboux and Halphen thus become applicable to this more general problem.

7. If in the problem of the motion of n bodies about a fixed center under forces varying as an arbitrary function of the distance we seek those integrals which do not involve the law of force,‡ we come upon a system of linear partial differential equations formed by equating to zero $n(2n+1)$ infinitesimal

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* Cyparissos Stéphanos: "Sur les forces donnant lieu à des trajectories coniques," *Crelle's Journal*, vol. 131 (1906), pp. 136–151.

† Darboux, Halphen, *Comptes Rendus*, vol. 84, pp. 760, 939.

‡ Gravé seems to have been the first to seek integrals of this kind in the problem of three bodies, and he found that the differential equations in the Bour-Bertrand form admit of no integrals independent of the law of force other than those already known. See his paper: "Sur le problème des trois corps," *Nouv. Ann. de Math.*, series 3, vol. 15 (1896), pp. 537–547.

transformations. That these infinitesimal transformations generate a continuous group was pointed out by Professor Lovett in an unpublished paper previously presented to the Society (December 29, 1902); in the present paper the methods of Sophus Lie are employed to construct the invariants of this group.

8. In Professor Hedrick's first paper an example of a surface is given in which the distance from a point on the normal to each normal section is a minimum, while the distance from the same point to the surface itself is not a minimum. The example shows the fallacy of a geometric conclusion sometimes assumed as obvious, and reference is made to a recent use of the fallacious theorem. (BULLETIN, volume 13, No. 9, page 447.)

9. In this note, which will be offered for publication to the *Annals of Mathematics*, Professor Hedrick gives an example of a closed curve formed by the junction of (an infinite number of) rectilinear segments, which is nevertheless *smooth*, in the sense defined by Professor Osgood (Funktionentheorie). The statement is to be understood in the sense that between any two points on the curve there is an arc which is a segment of a straight line. The example is obtained by simple integration from a well-known case; it seems worth while in illustrating the non-intuitive character of the ideas involved.

10. In Professor Carmichael's third paper a large class of transcendental functions is defined in the following manner: Let $u = f(x)$ and $v = f(y)$, and put $u^v = v^u$. This last equation has a solution $u = f(x) = \mu^{1/(\mu-1)}$, $v = f(y) = \mu^{\mu/(\mu-1)}$, where μ is a variable. Solving these functional equations, we have $x = g(\mu^{1/(\mu-1)})$ and $y = g(\mu^{\mu/(\mu-1)})$, where g is the function inverse to f. These equations define a function y of x. It is this function which is studied in the paper. Attention is confined chiefly to questions of continuity and development in series.

11. Lie established a one to one correspondence between the integrals of the canonical system of differential equations and the one-parameter continuous groups of contact transformations of which the system admits: *i. e.*, making use of an integral,

one can construct the infinitesimal transformation belonging to a group of which the system admits, and conversely by making use of the infinitesimal transformation one can construct an integral of the system.[*] This theorem is the foundation of the modern transformation theorems of dynamics.[†]

The single canonical substitution on the other hand has served chiefly in the simplification of the equations.[‡] The paper of Dr. Gillespie defines a single canonical substitution, the knowledge of which leads immediately to the construction of an integral of the system.[§]

13. Professor Miller's note aims to exhibit some elementary relations between well known methods of finding the primitive roots of a number and the properties of a cyclic group. It also contains a very elementary proof of the following theorem, which is believed to be new: When the exponent to which the primitive roots of a given odd number g belong is of the form $2q$, q being any odd prime, each of the primitive roots of g is given once and only once by $-\alpha^2$, $1 < \alpha < g/2$ and α being prime to g. When g is even, the primitive roots are obtained by replacing the even values of α^2 by $g/2 - \alpha^2$, while the odd values remain unchanged. In the former case α^2 belongs to exponent q, while α^2 or $g/2 + \alpha^2$ belong to this exponent for odd or even values of α, respectively, when g is even and has primitive roots. From this we derive the following useful corollary: Every prime of the form $2q + 1$, q being any odd prime, has for its primitive roots $-\alpha^2$, $1 < \alpha < q + 1$. It is also pointed out that the theorem which affirms that the product of all the primitive roots of any number which has more than one such root is congruent to unity is a special case of the theorem that the product of all the operators of the same order > 2 in any abelian group is identity.

15. Professor McMahon's paper considers the field of a vector whose three components at any point are the partial deriva-

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[*] Whittaker, Analytical dynamics, p. 308.

[†] l. c., p. 292.

[‡] Jacobi, Vorlesungen über Dynamik. Poincaré, Mécanique céleste, vol. 1. For condition that a substitution be canonical, see Lie, *Norw. Ark. für Math.*, etc., 1877. Also Jacobi, Vorlesungen über Dynamik.

[§] For the two theorems upon which the argument depends, see the author's thesis, "Anwendungen des Unabhängigkeitssatzes auf die Lösung der Differentialgleichungen der Variationsrechnung," p. 30; Göttingen, 1906. Also, "On the construction of an integral of Lagrange's equations in the calculus of variations," BULLETIN, vol. 13 (1907), pp. 345–348.

tives of a given function of the coordinates of the point. Among the topics treated are the following: Derivative of the vector for any direction in its field; variation of derivative; variation of tensor; rotation of the field vector for displacement in any direction; variation of rotation; lines of no rotation; family of surfaces orthogonal to the vector lines; relation of the indicatrix conic at any point to the vector field around the point; invariants of the field.

16. In the second paper by Professor Dickson it is shown at the outset that every maximum (*i. e.*, not contained in a larger) commutative group M of m-ary linear homogeneous transformations (a_{ij}) in a field F may be completely defined by linear homogeneous relations between the a_{ij} with constant coefficients belonging to F. In other words, the independent parameters of M enter linearly and homogeneously. Continuous commutative linear groups with the last property have been considered by Professor W. Burnside, *Proceedings of the London Mathematical Society*, volume 29 (1898), pages 325-352, chiefly from the standpoint of the invariant factors of the characteristic determinant of the general transformation of the group. He treats in detail the case of two invariant factors distinct from unity; but (page 341) draws an erroneous conclusion (that $b_1 = 0$, whereas $a_1 - e_1$ is zero in view of the initial hypothesis), which renders invalid the first step, as well as all the successive steps, of his reduction process. Moreover, of the groups thus omitted, there occur examples with $b_1 \neq 0$ which are not reducible to groups with $b_1 = 0$ and of the type considered. Burnside starts with a particular transformation S (in canonical form) with the desired invariant factors, then exhibits the most general linear transformation T commutative with S, and finally attempts to select and normalize all the transitive commutative groups composed exclusively of transformations T. He expected to prove that all such groups have the same invariant factors as the individual transformation S. It is clear *à priori* that this will be true only in trivial cases. A correct version of Burnside's methods of constructing commutative linear groups would therefore still have the objection of being highly redundant. To establish the non-equivalence of two groups, Burnside makes use (§11) of another classification (§9), based upon a highly desirable normal form of the commutative linear group. In this normal form, at least one variable x_i is merely

multiplied by a constant, at least one y_j is replaced by $by_j + \Sigma a_i x_i$, etc. As Burnside omits references, it should be said that this normal form was given implicitly by Lie in 1878 (cf. Lie-Engel, Transformationsgruppen, I, page 589), and explicitly for commutative linear groups of order a power of p in the $GF[p^r]$ by Jordan in 1870 (Traité des substitutions, page 147). However, the latter author makes errors of various kinds (on page 148) in determining the commutative groups of the simplest types, as is pointed out in detail in the present paper. By means of the Jordan normal form (established for a general field F) and the theorem stated at the outset, it is a comparatively simple algebraic problem to determine explicitly all the maximum commutative linear groups. The more difficult problem of their reduction to non-equivalent types is found to depend upon the normalization of families of quadratic forms $\Sigma c_i q_i$ by linear transformation both of the parameters c_i and the variables in q_i. The paper will be offered for publication in the *Annals of Mathematics.*

17. The third paper by Professor Dickson gives a direct elementary proof of Jordan's canonical form of any linear homogeneous transformation (Traité des substitutions, pages 114–126, for the case of integral coefficients reduced modulo p, a prime), as extended to the general case of coefficients in an arbitrary field F (*American Journal*, 1902, pages 101–108). Let the characteristic equation have the roots $\alpha, \beta, \cdots$, with the multiplicities $a, b, \cdots$. The plan of the proof is to introduce a variables depending on α, b variables depending on $\beta, \cdots$, and to show that the new variables are independent. As the various sets are here introduced independently and not in succession, one avoids the necessity of the numerous normalizations made by Jordan (pages 120–123) and the intricate proof (page 121) that each set involves a single irrationality. The essential value of the canonical form lies less in the form of the explicit formulas than in the exact relation of the new variables to the irrationalities introduced. The paper will be offered for publication in the *Annals of Mathematics.*

18. In this note Professor Kasner separates the geometric from the analytic elements involved in Lie's well-known interpretation, and obtains a concrete representation by introducing a surface whose contour lines are the curves defined by the

differential equation. The slope of this surface visualizes the integrating factor.

19. In a former paper Professor Kasner showed that when a surface is represented conformally on a plane its geodesics are pictured by a family of curves whose central loci are straight lines. In the case of curves of given geodesic curvature it is shown that the corresponding loci are conics. When the curvature is varied the conics corresponding to a given point have a common focus and a common directrix.

20. With reference to a remark made by Professor Study in the April, 1907, number of the *American Journal of Mathematics* (page 101, note) Mr. Schweitzer observed that he has already shown that right-handedness is a descriptive relation which, with the aid of the indefinable point and ordered dyad, generates a descriptive geometry extensible, on the basis of the axioms, to the usual spatial projective geometry by well known methods.[*]

Mr. Schweitzer then briefly indicated some of the more important results of his researches in descriptive, projective and metrical (euclidean) geometry, which will be offered to the *American Journal of Mathematics* for publication in detail:

(1) There exists not only a fundamental linear relation (viz., the relation of Vailati) but a fundamental n-dimensional descriptive relation R_n $(n = 1, 2, 3, \cdots)$. For $n = 3$ the relation is that of absolute right-handedness. Further, the relation R_n is n-dimensionally transitive, (*i. e.*, $xR_na_2 \cdots a_{n+1}$, $a_1R_nxa_3 \cdots a_{n+1}$, $\cdots a_1R_na_2a_3 \cdots x$, imply $a_1R_na_2 \cdots a_{n+1}$) and alternating.

(2) Descriptive or projective order in n-dimensions $(n = 1, 2, 3, \cdots)$ may be generated p-dimensionally $(p = 1, 2, 3, \cdots n)$, so that the geometric spaces may be said to have the double index (p, n), $1 \leqq p \leqq n$. The preceding results have an important critical bearing on the discussion of geometric order given by Russel in his Principles of Mathematics.

(3) The descriptive n-dimensional systems generated by the relation R_n may be transformed into equivalent systems generated by the relation K_n,[†] the latter for $n = 3$ express-

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[*] The relation of right-handedness referred to is the relation R_3 of the author, cf. BULLETIN, Nov., 1906, p. 79. The results indicated above were also demonstrated by the author in a paper read at the University of Chicago, Feb., 1905; at that time an undefined *class*, corresponding to the relation R_3, was used.

[†] Cf. BULLETIN, June and Nov., 1906.

ing concretely sameness of sense, or relative right-handedness. For $n \geqq 3$ Mr. Schweitzer showed that the corresponding n-dimensional K_n systems may be made "complete" for euclidean geometry by adding a certain number of axioms, the system of real numbers being introduced in a very simple manner. These "complete" systems constitute an elegant basis for Grassmann's Ausdehnungslehre and the exposition of the latter by Peano. In this connection it is of interest to consider a K_0 system of dimensionality zero, in addition to the K_n systems mentioned above.

21. In an earlier paper (read December 28, 1906), Dr. Griffin showed that Newton's law of attraction is the only one which permits the ellipse as a central orbit, and which satisfies the conditions: (1) The force is a single-valued function of the distance; (2) real motion is possible in all parts of the plane; and (3) there is no circle about the center of force, throughout which the force continually decreases toward the center. It has since been found that condition (1) is superfluous, as any law not satisfying (1) fails also to satisfy (2). In the present paper proof of this redundancy is given; and as a corollary thereto is obtained the theorem of Darboux and Halphen. *

22. It has been generally overlooked that the law of force, under which a given curve is described as a central orbit with a given constant of areas, cannot be uniquely determined from a knowledge of the position of the center of force. Dr. Griffin, treating this question in his second paper, considers especially a few of the infinitude of laws, other than that of Newton, for which a possible orbit is the ellipse with the center of force at a focus. Various properties of other trajectories than the original ellipse are pointed out.

23. This paper is the development of a communication made by Professor Davis to the Chicago section at its March meeting (see BULLETIN, June, 1907, page 436). In it are presented in order the notation with its adaptation to homogeneous coordinates and the effect of real and imaginary shifts of the axes; scalar and vector operations; the linear relation — central, non-central, and real; the quadratic relation as exhibited by the real and the imaginary circle and by other conics; the real cubic and its nine inflexions.

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* *Comptes rendus*, vol. 84, pp. 760–762, and 936–941. Cf. also Tisserand, Mécanique céleste, vol. 1, pp. 36–42.

24. In this second paper, Professor Davis applies his method to three-dimensional space. He considers the planar relation, axial, non-axial but spatial, non-spatial; the linear relation, spatial and axial, spatial but not non-axial; combination of linear and planar relations; the cylinder; the sphere; the connection with Von Staudt's representation of imaginaries by involutions.

25. Dr. Sisam showed that the quartic surface $\phi(A, B, C, D) = 0$, ϕ, A, B, C and D being quaternary quadratic forms, which was considered by Durrande * in the *Nouvelles Annales* as the most general quartic surface, is not the most general surface of fourth order, but that a single condition must be satisfied by the coefficients. He also showed that the equation of the most general quartic surface can be written in the form $\theta(A, B, C, D, E) = 0$.

26. In Professor Snyder's second paper it is shown that if a plane curve of order n has not more than $2(n-4)$ double points, the simplest curve to which it can be birationally reduced can be obtained by a Cremona transformation. The maximum number of basis points of a net of adjoint curves of order n can be found from any point P and its images under the transformation $x' = x, y' = \theta y, z' = \theta^n z$, $(\theta^{n^2-n+1} = 1)$. These theorems are applied to space curves, and it appears that the plane curves which are the projections of complete intersections of two surfaces of orders m, m' cannot be birationally transformed to curves of order less than $mm' - 1$.

28. In Professor Veblen's note are defined and studied what may be called continuous increasing functions of (finite and transfinite) ordinal numbers. They are such that x and $f(x)$ are both ordinals; if $x_1 < x_2$, then $f(x_1) < f(x_2)$; if $\{x'\}$ is the set of all ordinals less than an ordinal x of the second kind, then $f(x)$ is the least ordinal greater than every $f(x')$. A well-ordered set of these functions is constructed of which Cantor's exponential function ω^x is the first. The study of this set of functions leads to a classification of Cantor's ϵ-numbers.

29. Lüroth's quartic is characterized by containing the ten intersections of five secant lines; in fact it has an infinite system

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* *Nouvelles Annales*, series 2, vol. 9, p. 410.

of inscribed quinquilaterals, whose lines are tangent to a conic. It is associated to a determinate Clebschian quartic which has every such quinquilateral for a polar polygon. Each of these quartics is characterized projectively by a single invariant condition; that of the Clebschian is a known simple invariant, but for the Lürothian the condition has not been worked out. Miss Miller and Professor White show that it can be expressed by equating to zero the resultant of six cubics homogeneous in six unknowns.

This form of quartic is discussed by Lüroth in volumes 1 and 13 of the *Mathematische Annalen*; at the end of his second article he speaks particularly of the present problem. Scorza, in volume 2 (third series) of the *Annali di Matematica*, and in volume 52 of the *Mathematische Annalen*, makes an exhaustive study of the relation between the Lürothian and the Clebschian by the use of the 36 even thetas of three arguments. A short paragraph in Wieleitner's Theorie der ebenen algebraischen Kurven höherer Ordnung treats Lüroth's curve geometrically.

30. In the *Transactions* for January, 1906, and January, 1907, Professor Fite discussed the manner in which the degree of an irreducible linear homogeneous group is determined by the abstract properties of the group. The discussion was limited for the most part to groups whose orders are powers of a prime. In the present paper the same question is considered for a somewhat different category of groups — namely, those in which every non-invariant commutator gives an invariant commutator besides identity. It is shown that the degree of such an irreducible group is uniquely fixed and an expression for the degree is given.

Besides the articles by the author already cited, reference may be made to Frobenius, "Ueber die Primfactoren der Gruppendeterminante," *Berliner Sitzungsberichte*, 1896, II, page 1343, and Burnside, "On the reduction of a group of homogeneous linear substitutions of finite order," *Acta Mathematica*, volume 28, page 369.

31. Given a linear homogeneous substitution of degree n and period m, whose coefficients are rational numbers, and such that there is no similar substitution of degree less than n, then n is a certain definite number-theoretic function of m. This function enables Dr. Ranum to determine all the finite periods of

n-ary linear substitutions having rational coefficients. A linear substitution is said to be reducible in a given field if it can be transformed into a reduced form, whose coefficients belong to the field. In the field of rational numbers it is found that every irreducible linear substitution of period m is of degree $\phi(m)$. By means of this and other considerations, all n-ary linear substitutions of period m with rational coefficients are easily classified as to their reducibility; finally a method is given of finding a rational canonical form of every such linear substitution.

32. Professor Stephens showed that certain curves, whose equations are of the form

$$t^n - xt^{n-1} + \mu t^{n-2} - \alpha\mu t^2 + \alpha\mu t - \alpha = 0$$

are easily constructed mechanically — the instrument used being similar in principle to that for the pentadeltoid. These curves are of class n and order $2(n-1)$. Each curve is circumscribed to an ellipse, touching it in n points. In any direction there are $n-2$ parallel tangents, the centroid of whose points of tangency is constant. The line at infinity is a double tangent. The cusps are n in number, real or imaginary. The special case $\mu = (n-1)/(n-3)$ exhibits some interesting properties.

33. The possible groups to which algebraic configurations of genus 4 belong have been determined by Wiman,[*] by means of collineations in space of three dimensions. Miss Van Benschoten completes Wiman's results by interpreting geometrically the various transformations when the configuration is expressed as a plane binodal quintic curve, and by deriving their equations. Every curve c_5^4 of order 5 and genus 4 is the plane projection of a space sextic R_6 lying on a quadric F_2 and a system of cubic surfaces which in particular cases may become cones K_3. An illustration is furnished by the octahedron group; it leaves a plane and its pole as to F_2 invariant. The six points of R_6 in this plane are points of tangency of lines through the pole. R_3 lies on 6 K_3, whose vertices form a complete quadrilateral in the invariant plane. The points of R_6 lie on the sides of the diagonal triangle; they are also in six-fold involution on

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[†] Wiman, "Ueber die algebraischen Kurven von den Geschlechtern 4, 5, 6," Stockholm Academy, *Bihang till Handlingar*, vol. 21 (1895).

the conic containing them. This configuration completely determines the axial and central involutions which leave the curve invariant, and indirectly defines the group. After deriving this and similar results for the other groups, the R_6 is projected into c_5^4 and the corresponding forms discussed.

34. In the problem of minimizing the integral

$$I = 2\pi \int y \sqrt{x'^2 + y'^2}\, dt$$

between the two fixed points A_0 and A_1, Miss Sinclair proves that either the catenary or the Goldschmidt discontinuous solution furnishes an absolute minimum with respect to rectifiable curves in the region $y \geqq 0$. She then discusses a discontinuous solution consisting of three curves which meet in a point A_2, a catenary through A_0 and A_2, a second catenary through A_2 and A_1, and a normal to the x-axis. She finds that with respect to similar systems of rectifiable curves in the region $y \geqq 0$ the above extremal system furnishes a relative minimum for the integral under the following necessary and sufficient conditions:

1) The vertices of the two catenaries are equidistant from the x-axis, and meet each other and the vertical line through A_2 at an angle of 120°.

2) $A_0(t = t_0)$ is limited in position by the inequality $t_0 > t_0^*$, where t_0^* is the unique negative solution of the equation $\coth t - t = 2 + \log \sqrt{3}$, and $A_1(t = t_1')$ is limited by the inequality $t_1 < t_0'$, where t_0' is the unique positive solution of the equation $\coth t - t = \coth t_0 - t_0 - \log 3$, and $A_0'(t = t_0')$ is the conjugate of A_0, and is the point in which the system touches the envelope of the set of extremal systems through A_0 which satisfy condition 1). A_0' may also be located by a Lindelöf construction. The value of I furnished by this discontinuous solution is for a certain region less than that given by the Goldschmidt solution, but never less than that given by the single catenary through A_0 and A_1. The system may be constructed by means of liquid films. Actual measurements of the limit of stability of the film correspond nicely to the theoretical values for the conjugate point.

35. The first aim of Dr. Fréchet's paper is to generalize the known results concerning linear functional operations (and particularly Hadamard's theorem) to cover the case where these

operations are defined not only, as before, in the field of continuous functions (of a single variable) but also in the far greater field of the functions which have an integral according to Lebesgue's definition. The second point is to prove that any of the operations defined in such a field can be determined by one continuous function $u(x)$ of a single variable, called characteristic function of the operation. The operation U_f is numerically known by means of the limit of one double integral when $f(x)$ and $u(x)$ are known. Conversely $u(x)$ is obtained by applying the operation to the function $\phi_y(x)$, $\phi_y(x)$ being equal to 1 when $0 \leqq x \leqq y$ and to 0 when $y < x \leqq 2\pi$.

36. The original object of Professor Greenhill's memoir was to provide the simplest method of calculating numerically the electromagnetic constant of a helix, in the ampère balance, designed by the late Viriamu Jones and Professor Ayrton for weighing the electromagnetic attraction and so arriving at an independent measure of the electrical units.

Incidentally it was requisite to coordinate the conflicting notation of various writers on the subject, by adopting Maxwell's Electricity and Magnetism as a standard, and to develop Landen's quadric transformation in order to reconcile results apparently discordant, so that the fundamental geometric meaning of Landen's transformation appears in the course of the electromagnetic theory.

The analytic complexity in the reduction of the elliptic integral in electromagnetism, as well as in most dynamical problems arises in consequence of the appropriate integral of the third kind being of the circular form in Legendre's terminology; the elliptic parameter of Jacobi is then a fraction of the imaginary period, so that the expression by the theta function implies a complex argument and a table of the theta function would not be of complete utility unless made in a triple entry form.

But in the practical problems of electromagnetism it is the complete third elliptic integral which is sufficient for a solution, and this, as Legendre has shown, can be expressed by integrals of the first and second kind, complete and incomplete, for which Legendre's Table IX provides the numerical value.

In working with the third elliptic integral it is found a practical convenience to retain it in the algebraical form, and to delay the adoption of a Legendrian, Jacobian, or Weierstrassian notation; the circular form of the complete integral will then fall into one of four classes, according to the sequence of the quantities involved.

The geometric interpretation of a formula has been explained wherever possible, when used for the expression of an electromagnetic quantity, as mutual induction, magnetic potential, vector potential, and so forth; and the advantage of the Stokes function in analytic simplicity has been emphasized over the ordinary potential function.

F. N. COLE,
Secretary.

ON A SPECIAL ALGEBRAIC CURVE HAVING A NET OF MINIMUM ADJOINT CURVES.

BY PROFESSOR VIRGIL SNYDER.

(Read before the American Mathematical Society, September 5, 1907.)

IN researches on plane curves a very fruitful configuration has been employed by Küpper* in obtaining particular curves having double points in abnormal position. His method consists in using as basis points part of the intersections of two curves which satisfy certain prescribed conditions; pencils and nets are then constructed having these basis points as nodes. The question naturally arises whether the same procedure can be employed in other cases, making use of all the constants in the system. In the following note it will be shown that in such cases the special series which are obtained cannot be employed to reduce the order of the curve. Incidentally, an illustration is furnished of configurations having a $g^2_{2(n-1)}$ (a linear series with two degrees of freedom and of order $2(n-1)$), although the curve cannot be reduced to order $2(n-1)$.

1. Through $n-1$ points on a straight line pass two general curves of order n, c_n, c'_n. These curves intersect in n^2-n+1 residual points through which can be passed ∞^2 curves of order n. Any two points P, Q of the plane will determine two pencils $c_n+\lambda c'_n=0$, $\phi_n+\lambda\phi'_n=0$ contained in this net, which can so be made projective that the locus of the variable intersections is the curve c_{2n} of order $2n$

$$(1) \qquad c_n\phi'_n-c'_n\phi_n=0.$$

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*C. Küpper: "Ueber das Vorkommen von linearen Schaaren . . . ," *Sitzungsberichte der Böhmischen Gesellschaft*, Prag, 1892, pp. 264–272.

The $n^2 - n + 1$ basis points are all double points of c_{2n}; the net c_n defines upon it a $g^2_{2(n-1)}$. If c_{2n} has no other double points, the net defines an adjoint system. The canonical adjoint curves of order $2n - 3$ can be composed of ϕ_n and any fixed curve of order $n - 3$. The usual form of the theorem of birational transformation is that any system of adjoint curves of order $2n - 3$ and having two degrees of freedom may be used for transforming curves, but for our net the double points furnish the minimum number of conditions for a curve of order n. Moreover, the $g^2_{2(n-1)}$ is of such a nature that one point determines $n - 2$ others, analogous to the case of the hyperelliptic curves. If $n > 3$, $p > 2n - 2$, hence c_{2n} cannot be hyperelliptic.

2. When $n = 3$, the g^2_4 is composed of two g^1_2, the groups being associated arbitrarily, hence it is hyperelliptic.

The necessary and sufficient condition that a sextic curve with seven double points be hyperelliptic is that every pencil of cubic curves through the seven double points and one simple point on the curve shall have the residual basis point on the curve also.

The curve can be reduced to a quintic with a triple point by means of adjoint quartics; it cannot be reduced to a quartic curve.

3. If we now write $\rho x_1 = c_n\chi_{n-3}$, $\rho x_2 = c'_n\chi_{n-3}$, $\rho x_3 = c''_n\chi_{n-3}$ then (1) will go into a conic section, but it will be shown that (1) can never be rational. If we use (for a moment) the notation of Amodeo in his memoir on minimum adjoint curves (*Rendiconti dei Lincei*, series 5, volume 2 (1893), pages 460–467) we have $m = 2n$, $p = n^2 - 2n$, $\alpha = b - 3$, $\rho = \frac{1}{2}(n - 2)(n - 3)$, wherein adjoint curves of c_m of order $m - 3 - \alpha$ exist and will pass through the remaining ρ double points if required to pass through all but ρ. As is shown by Amodeo, if these conditions are satisfied, then the intersection of c_{2n} with any c_{n-3} (not adjoint), forming a $g^{n/2(n-3)}_{2n(n-3)}$, will not be a complete series, but will be contained in another having ρ further degrees of freedom. Equation (1), page 466, is satisfied if $t = 1$, hence all the formulas are satisfied when our c_{2n} has the maximum genus $n^2 - 2n$.

4. Any two curves of the net will intersect in $n - 1$ points lying on a straight line. The pencil of adjoint curves which has any fixed basis point on c_{2n} will have $n - 2$ other fixed basis points upon it, and define a g^1_{n-1}. The $n - 1$ points of each group of this series are collinear. Such a series can be defined by starting from any point of c_{2n} as basis point, but from

the form of the equation it immediately follows that *all the* g^1_{n-1} *are identical.* The groups of g^1_{n-1} determine an $[n-2]$ involution on c_{2n}. If p be the genus of c_{2n}, the number of coincidences is $2(n+p-2)$.*

If c_{2n} has a double point apart from the n^2-n+1 basis points, it will count for two coincidences, hence if c be the number of simple coincidences we have

$$2(n+p-2) = 2(n^2-2n-p)+c$$

or

$$4p = 2n(n-3)+c+4.$$

Since c cannot be negative, the minimum value of p is that of the general c_n, thus

$$\tfrac{1}{2}(n-1)(n-2) \leqq p \leqq n(n-2).$$

If singular G_{n-1} exist in g^1_{n-1}, double points may arise which do not involve two coincidences, but they can only occur for particular relations among the coefficients of c_{2n}, which we exclude.

Since $p > 0$, we can now state the following results:

(*a*) *The complete series* $g^2_{2(n-1)}$ *cannot be used for birational transformation.*

(*b*) *The envelope of the straight lines on which lie the groups of* g^1_{n-1} *is a conic.*†

(*c*) *The normal or canonical form of* c_{2n} *is a conic, taken* $(n-1)$ *fold.*

The analogy between these curves and the hyperelliptic curves thus becomes apparent.

5. When $n=4$, it was seen that the straight lines of the plane cut from c_8 a partial series g^2_8 contained in the complete series g^3_8, hence c_8 is the projection of a space curve of the same order. Curiously, the same thing is true for every value of n. Consider the space curve on the quadric surface, cutting the generators of one system in $n+1$ points, and those of the other in $n-1$ points. The space curve R_{2n} has therefore $h=n^2-n+1$

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* From the Cayley-Brill principle of correspondence, or from Zeuthen's formula. See Bertini: "La geometria delle serie lineari sopra una curva piana secondo il metodo algebrica," *Annali di Mat.* (2), vol. 22 (1894), pp. 1–40.

† Amodeo, "Curve κ-gonali," *Ann. di Mat.* (2), vol. 21 (1893), pp. 221–336. See formula, bottom page 231. $m=2n$, $\kappa=n-1$, $t=1$, hence y, the class of the envelope, is 2.

apparent double points. If now we project into c_{2n} from a center not on the quadric, the $n^2 - n + 1$ double points will be distinct. We thus have the theorem: "If through $h - \frac{1}{2} i(i + 1)$ of the h double points of the projections curve c_{2n} a curve of order $2n-i-3$ can be passed ($2n-i-3 \geqq n-1$), then this curve will pass through the remaining $\frac{1}{2} i(i + 1)$ double points also."*

Through $\frac{1}{2} n(n + 3) - 2$ arbitrary points can be passed ∞^2 c_n. If $i = n - 3$, we see from the above theorem that all such curves c_n through $\frac{1}{2} n(n + 3) - 2$ of the double points will pass through the others also, hence a net of adjoint curves of order n exists for our projection curve c_{2n}. The adjoint curves define upon c_{2n} a $g^2_{2(n-1)}$. In case $n = 4$, the g^2_6 is the residual of g^3_8 by the Riemann-Roch theorem, but in no case can this net be used to transform the curve c_{2n} birationally into another of order $2(n - 1)$. If we project R_{2n} from a point of the quadric on which it lies, c_{2n} will have an $(n - 1)$ fold point P_{n-1} and a P_{n+1}, and in general no other singularities. The g^1_{n-1} defined by one system of generators is thus defined by one pencil of lines, and the g^1_{n+1} by another. By projecting R_{2n} from a point upon the curve, or by ordinary inversion of c_{2n} we obtain a c_{2n-1} with a P_n and a P_{n-2}, and no other singularities. By inversion, an infinite number of projectively distinct c_{2n-1} can be obtained, but they will all have multiple points of the same order. In no case can this curve be reduced to a curve of lower order by any birational transformation.†

When $n = 4$, c_8 has a g^1_3 which is invariant, but when c_m has a g^1_3, $p \leqq 2m - 5$; since c_8 is of genus 8, it follows that $m \geqq 7$.‡

Probably similar deductions can be made for larger values of n from the memoirs of Amodeo and of Bertini. Any adjoint ϕ_n containing a point P must also contain the whole group G_{n-1} to which P belongs (Bertini, l. c., page 29), hence if two different ϕ_n pass through P, the $n - 1$ points of intersection will be the G_{n-1} to which P belongs. These points always lie on a straight line (Bertini, page 30, *b*). This shows that our projection curve c_{2n} can be generated projectively by the method of § 1, and the two systems are thus coextensive. Since on R_{2n} the g^1_{n-1} is defined by the generators of one system of the hyperboloid, *the envelope of the lines containing the* G_{n-1} *is the section of the tangent cone to the hyperboloid from the center of projection.*

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* H. Valentiner, "Zur Theorie der Raumkurven," *Acta mat.*, vol. 2 (1883), pp. 136–230. See p. 177. Also M. Noether, "Zur Grundlegung der Theorie der algebraischen Raumkurven," *Berliner Abhandlungen* (1883), pp. 1–120. See p. 24.

† V. Snyder. "On birational transformation of curves of high genus," *Amer. Jour. Math.*, vol. 30 (1908), pp. 11–20.

‡ K. Bobek: "Ueber Dreischaarkurven," *Wiener Berichte*, vol. 98 (1889), pp. 141–172.

6. If ϕ_n, ϕ'_n, ϕ''_n be three cones of order n containing the n bisecants from (0, 0, 0, 1), then, since the $n-1$ remaining edges of intersection with the cone R_{2n} from the same point lie in a plane, the equation of the defining monoid may be written

$$w = \frac{x\phi'_n}{\phi_n} = \frac{x\phi''_n}{\phi'_n},$$

from which the equation (1) results. Incidentally, these equations furnish a means for reducing c_{2n} to c_{2n-1}, namely, the ∞^2 plane sections of the monoid from a point on R_{2n}, lying on one of the $n-1$ simple edges.

7. If R_{2n} has also actual double points or cusps, ϕ_n will not in general pass through them, hence in the plane curves c_{2n} we can distinguish between projection of actual double points and apparent double points. Actual double points will not always absorb two coincidences in the $[n-2]$ involutions, but when $p < \frac{1}{2}(n-1)(n-2)$, the projection curve can not be written in the form (1).

For other curves on the hyperboloid, the maximum number of basis lines of a net formed by bisecants will not be reached; but when no actual double points occur we may say that the projection curve c_n with $p > \frac{1}{2}(n-1)(n-2)$ cannot be birationally transformed into any curve of order less than $n-1$.

CORNELL UNIVERSITY,
August, 1907.

NOTE ON CERTAIN INVERSE PROBLEMS IN THE SIMPLEX THEORY OF NUMBERS.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, September 5, 1907.)

Legendre* has considered the problem of finding the highest power of a prime p contained in $m! = 1 \cdot 2 \cdot 3 \cdots m$. Let m be written in the form

$$(1) \qquad m = a_0 p^\alpha + a_1 p^\beta + a_2 p^\gamma + \cdots,$$

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* A. M. Legendre, Théorie des nombres, 3d ed., I., p. 10.

where $\alpha, \beta, \gamma, \cdots$ are different positive integers (including zero) and $a_0, a_1, a_2, \cdots$ are the same or different positive integers each less than p. Put

$$(2) \qquad a_0 + a_1 + a_2 + \cdots = s.$$

Then the highest power of p contained in $m!$ is $p^{(m-s)/(p-1)}$; or, in the congruence

$$(3) \qquad m! \equiv 0 \pmod{p^{(m-s)/(p-1)}},$$

the modulus gives the highest power of p for which the congruence holds.

The problem first to be considered in this note is to find the solutions of (3) when p and s are known and m is the unknown.[*]

Suppose a solution

$$(4) \qquad m_1 = a_0 p^\alpha + a_1 p^\beta + a_2 p^\gamma + \cdots$$

has been found. Take α to be the least of the numbers $\alpha, \beta, \gamma, \cdots$ and divide equation (4) by p^α. We obtain

$$(5) \qquad \frac{m_1}{p^\alpha} = a_0 + a_1 p^{\beta-\alpha} + a_2 p^{\gamma-\alpha} + \cdots = m_2 \text{ say}.$$

It is clear from the first paragraph that m_2 is also a solution of (3). Moreover if m_2 is multiplied by any power of p the resulting number is also a solution of (3), subject to all the conditions which have been imposed. Therefore,

To every solution of (3) *there corresponds an infinite number of solutions differing only by factors which are powers of* p.

We shall define a characteristic solution to be such a one as that given in (5); thus, *a characteristic solution of* (3) *is one for which* m *is not divisible by* p.

Now suppose s separated in any way into parts each of which is less than p and let these parts be $a_0, a_1, a_2, \cdots$. Then (1) is a solution of (3) however $\alpha, \beta, \gamma, \cdots$ may be chosen so that

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[*] The modified problem of finding the solutions of the congruence

$$m! \equiv 0 (\bmod p^v),$$

subject to the limitation that p^v is the highest power of p in $m!$, is very easy. It is clear that the smallest solution is always a multiple of p, say $= \mu p$. Then other solutions are $\mu p + 1, \mu p + 2, \cdots, \mu p + p - 1$; moreover these are all the possible solutions. It is to be noticed that a solution of this problem does not always exist. A case in point is $p = 3$, $v = 3$; for these values there is no solution of the congruence subject to the imposed conditions.

they are different positive integers including zero. Hence, when $s > 1$ so that this partition is possible, *there is an infinite number of solutions for each partition of s as above.* When $s = 1$ it is clear that the solutions are $m = 1, p, p^2, \cdots$. Furthermore it is evident that there can be no solutions except those which are here obtained by aid of the partitions of s into parts each less than p.

As a second problem we shall consider the solution of

$$(6) \qquad m! \equiv 0 \ (\mathrm{mod}\ p^{m-t}),$$

where p^{m-t} is the highest power of p contained in $m!$, m still being the unknown. For every solution m we have from the proposition of the first paragraph

$$(7) \qquad m - t = \frac{m - s}{p - 1};$$

whence, if $p \neq 2$, that is, if p is an odd prime,*

$$(8) \qquad m = \left(1 + \frac{1}{p-2}\right) t - \frac{s}{p-2},$$

where s as before equals $a_0 + a_1 + a_2 + \cdots$, the sum of the coefficients of the powers of p when m is expressed in the form of (1). Therefore, since

$$(9) \qquad m < \left(1 + \frac{1}{p-2}\right) t,$$

a definite constant, *the number of solutions of this problem is finite.*

Now suppose

$$m = c_n p^n + c_{n-1} p^{n-1} + \cdots + c_1 p + c_0.$$

Then

$$s = c_n + c_{n-1} + \cdots + c_1 + c_0.$$

Hence from (8) we may write

$$(10) \qquad \begin{aligned} c_n p^n + c_{n-1} p^{n-1} + \cdots + c_1 p + c_0 + \frac{c_n + \cdots + c_1 + c_0}{p-2} \\ = \left(1 + \frac{1}{p-2}\right) t = r_\nu p^\nu + \cdots + r_1 p + r_0 + \frac{u}{p-2} \end{aligned}$$

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* If $p = 2$, $t = s$; this problem therefore reduces to the preceding for the case $p = 2$. Its solution has already been found. We shall henceforth consider p an odd prime.

where m and s are replaced by their values above and $r_\nu, \cdots, r_0$ are integers each less than p and $u < p-2$. The r's are uniquely determined from the known value of $[1+1/(p-2)]t$. Now

$$(11) \quad \frac{c_n + \cdots + c_0}{p-2} \leqq \frac{(n+1)(p-1)}{p-2} < p^n, \text{ if } n > 1.$$

Therefore in equation (10) $n = \nu$, when $n > 1$; then also $c_n = r_\nu$. Having obtained these values we may proceed similarly to consider the coefficeints c_{n-1}, $r_{\nu-1}$ and so on. All the possible solutions will be obtained with but little difficulty. The work will be made easy from the fact that the coefficients of like powers of p are the same in the two members of (10) except for small exponents of p.

As an example, consider the solution of the problem when $p = 11$ and $t = 14578$. We have

$$\left(1 + \frac{1}{p-2}\right) t = 16197\tfrac{7}{9} = 11^4 + 11^3 + 11^2 + 9 \cdot 11 + 5 + \tfrac{7}{9}.$$

Comparing with (10) we have $n = 4$, $c_4 = 1$. Then without difficulty we may further show that $c_3 = 1$ and $c_2 = 1$. Substituting in (10) and reducing we have

$$c_1 \cdot 11 + c_0 + \frac{c_1 + c_0}{9} = 9 \cdot 11 + 5 + \tfrac{7}{9}.$$

From this equation it is readily shown that $c_1 > 8$ and < 10; and therefore its only possible value is $c_1 = 9$. This gives $c_0 = 4$. Therefore,

$$m = 11^4 + 11^3 + 11^2 + 9 \cdot 11 + 4 = 16196.$$

There is nothing in the preceding discussion to show that a solution always exists for this problem. An examination of the product of the natural numbers in order, reference being had to how often any prime has entered up to any number m, will easily lead to the discovery that the consecutive values of t contain every integer beginning with 1 and that some integers will be repeated in the series of values of t. Hence, *there always exists at least one solution; and in some cases there is more than one solution.*

ANNISTON, ALA.,
May, 1907.

THIRD REPORT ON RECENT PROGRESS IN THE THEORY OF GROUPS OF FINITE ORDER.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, September 6, 1907.)

THE following general treatises have appeared since the presentation of my second report* a little more than five years ago: The constructive development of group theory with a bibliography, by B. S. Easton, 1902; A brief work in Russian entitled The polyhedron groups, by G. W. Pfeiffer, 1903; Mathematical crystallography and the theory of groups of movements, by H. Hilton, 1903; Eléments de la théorie des groupes abstraits, par J. A. de Séguier, 1904; and Quelques considérations sur les groupes d'ordre fini et les groupes finis continus, par Raymond Le Vavasseur, 1904. The number of treatises devoting at least one chapter to our subject has been considerably increased. Two have appeared in America: Dickson, Introduction to the theory of algebraic equations, 1903, and Cajori, An introduction to the modern theory of equations, 1904.

Unfortunately the report by Dr. Steinitz, which was announced by the Deutsche Mathematiker-Vereinigung at different times had to be abandoned. The works by Loewy and Wiman entitled respectively Vorlesungen über die Theorie der linearen Substitutionsgruppen and Endliche Gruppen linearer Transformationen, which were expected to appear in Teubner's Sammlung, have also been delayed. On the other hand Netto is preparing an elementary work for the Sammlung Schubert under the title Gruppen und Substitutionentheorie, and the second part of de Séguier's work is in press.

In view of the very large number of new theorems it appears impracticable to attempt to enumerate all of them. In the case of the larger memoirs where a number of connected theorems are developed I can generally only give such references as indicate the nature of the main questions, while the theorems which have appeared in very brief articles or in less accessible places receive relatively more attention. Moreover, it did not appear best to crowd the report with a complete list of references, since bibliographical works are more convenient for the purpose of learning all the details along special lines. I hope to have succeeded in exhibiting the connections between some isolated developments which are of importance, and that the report may thus arouse a wider interest along certain lines. It may be added that the advances in the theory of abstract groups have been noted in greater detail than those which come under the other headings of this report.

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* BULLETIN, November, 1902, p. 106.

§ 1. Elementary Applications.

Among the most elementary and the most important applications of groups of finite order are those which relate to the fundamental operations of arithmetic. The addition and the multiplication groups[*] were developed principally during the periods covered by the preceding reports. One of the most important extensions of the theory of the latter, made during the period covered by the present report, is given by the following theorem: The necessary and sufficient condition that a given set of numbers, which are distinct with respect to modulus m, generates a group with respect to multiplication is that the highest common factor of the modulus and a number of the set is independent of the choice of this number and that the quotient obtained by dividing m by this highest common factor is prime to this factor.[†]

If x_1 and n represent any two numbers (real or complex), the number $x_1 - n$ can be obtained geometrically by reflecting the point representing n, in the complex number plane, on the middle point of the segment connecting x_1 with the origin; that is, by connecting the point representing n with this middle point and extending the segment to an equal distance beyond this point. From this standpoint subtraction may clearly be regarded as an operation of period 2; and, if n is subtracted from a series of numbers $x_1, x_2, \cdots, x_l$ and the remainders are again subtracted from these numbers, in any order, these operations of period 2 generate a dihedral group and every dihedral group may be generated in this way. It is of interest to observe that the series of numbers $x_1, x_2, \cdots, x_l$ may be replaced by two numbers without affecting this dihedral group,[‡] but a further reduction is generally impossible. As the group is dependent merely upon the difference of these numbers and the modulus it is clear that every dihedral group may be generated in an infinite number of ways as a subtraction group.

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[*] BULLETIN, vol. 7 (1900), p. 122; *Annals of Mathematics*, vol. 2 (1901), p. 77.

[†] *Annals of Mathematics*, vol. 6 (1905), p. 44.

[‡] Loc. cit., p. 43.

The applications of these groups which have been most fully developed are those connected with the notions of complement and supplement of an angle in elementary trigonometry. As the modulus in this case is 360°, this group is equivalent to the one obtained by subtracting from 1 and 2 with respect to modulus 4. Hence it is identical with the group of movements of the square. The fact that there are two sets of four angles which do not go into eight distinct angles by means of the operations of taking the complement and the supplement is thus related to the general theory that, with respect to the subtraction group, there are always two and only two such sets of numbers, and the total number of different numbers in the two sets is equal to the order of the dihedral group. The more general applications of the subtraction groups in elementary trigonometry have also been considered and a few illustrative examples are given in a recent article in the *Annals of Mathematics.*[*]

Results of unusual interest relate to the groups generated by the two operations of subtracting from a fixed number and dividing either the same or a different number. These groups differ from those considered above with respect to the fundamental property that it is not necessary to assume a modulus in order to obtain finite orders. If the subtrahend is denoted by x_1 and the dividend by x_2, the two operations $x_1 - n$, x_2/n generate a dihedral group, since each of these operations is of order 2. The important result that only four distinct groups of finite order are possible when x_1 and x_2 are both rational numbers has been proved both analytically and geometrically.[†] The orders of these groups are 4, 6, 8, and 12. These orders are obtained by assuming $x_1 = 0$, $x_1^2 = x_2$, $x_1^2 = 2x_2$, and $x_1^2 = 3x_2$ respectively. There are three sets of conjugate numbers under such a group which include all the numbers which are transformed into a smaller number of distinct values than the order of the group.[‡]

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[*] Vol. 8 (1907), p. 97.

[†] *Quar. Jour. of Mathematics*, vol. 37 (1905), p. 80; Hilton, *Messenger of Mathematics*, vol. 35 (1905), p. 117.

[‡] *Quar. Jour. of Mathematics*, loc. cit.

If it is not assumed that x_1, x_2 are rational, it is possible to assign them values so that the two operations $x_1 - n$, x_2/n generate a dihedral group of any given possible order. One set of values which leads to the dihedral group of order 10 is $x_1 = 1 + \sqrt{5}$, $x_2 = 4$. Among the sets which give rise to the symmetric group of order 6, that in which $x_1 = x_2 = 1$ has received most attention, as this is the group formed by the six values of an anharmonic ratio as well as by the six trigonometric functions $\sin^2 \alpha$, $\sec^2 \alpha$, $-\cot^2 \alpha$, $-\tan^2 \alpha$, $\csc^2 \alpha$, $\cos^2 \alpha$. As the four-group is the only invariant subgroup (besides identity) of a symmetric group which gives rise to a quotient group whose order exceeds 2, the six values of an anharmonic ratio furnish a unique instance of a more than two valued rational function all of whose values are rationally expressible in terms of one of them. That is, any other function having this property is rationally expressible in terms of an anharmonic ratio.

Another instance of very elementary applications which were developed during the period covered by the present report is furnished by the groups of the figures of elementary geometry.* The group of a figure is composed of all the movements of space which transform the figure into itself. In these movements space is regarded as rigid in the sense that any two points are transformed into two points at the same distance from each other. Every movement which transforms a system of points into itself must transform the center of their minimum circumscribing sphere into itself. Hence such movements must be composed of rotations around this center, and all the groups of movements of a system of coplanar points, finite in number, must be either cyclic or of the dihedral rotation type. Moreover, the dihedral group of order $2n$ is the group of the regular plane polygon of n sides. In particular, the octic group is the group of movements of the square; while the rectangle with unequal sides has the transitive four-group for its group of movements, and the rhombus which is not a square has the intransitive four-group for its group of movements. If the base of a regular pyramid has $n > 3$ sides, its group is cyclic and of order n, and if the base of a regular prism has n sides $(n \neq 4)$, its group is dihedral rotation and of order $2n$.

During the last few years some of the direct applications of substitution groups in elementary trigonometry have received notice.* If a, b, c represent the sides of a spherical triangle, and α, β, γ the supplements of the angles opposite those sides in order, it follows from the properties of the polar triangle that the substitution $a\alpha \cdot b\beta \cdot c\gamma$ transforms any formula relating to the spherical triangle into one which is equally true. It is also evident that any formula which has been derived without specializing any one of the six parts in question is transformed by the two substitutions $ab \cdot \alpha\beta$, $bc \cdot \beta\gamma$ into one which is equally true. Hence it follows that the group of order 12 which is generated by these three substitutions transforms any such formula into those which are equally true. This well known imprimitive group of order 12 and degree 6 may therefore be employed to derive a system of formulas from a given one, and hence it may sometimes be used to advantage even in a very elementary course of instruction. Its intransitive subgroup of order 6 may be employed to some advantage in plane trigonometry.

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* *Amer. Math. Monthly*, vol. 10 (1903), p. 215.

§ 2. ABSTRACT GROUPS.

During the period under consideration great progress was made in the theory of abstract groups, just as had been the case during the preceding period. I will first state some of the new theorems relating to generational operators. If a series of operators is such that each of them transforms all the others either into their inverses or into their third powers, the totality of these operators generates the hamiltonian group of order 2^n, except when all of them are of order 2.† In this special case it generates the abelian group of order 2^m and of type $(1, 1, 1, \cdots)$. If a non-abelian group is generated by two operators which transform each other into their α, β powers respectively, the group is completely defined by these powers whenever the indices of the powers diminished by one are such that one of them is a prime while the other is a power of this prime, and only then. In other words, the non-abelian group of order p^m which contains a cyclic subgroup of order

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* *Quar. Jour. of Mathematics*, vol. 37 (1906), p. 226.

† From this theorem we may readily derive two very simple definitions of the quaternion group, viz., the quaternion group is the only non-abelian group generated by two operators which transform each other into their third powers, and the quaternion group is the only non-abelian group generated by two operators which transform each other into their inverses. *Prace Matematyczno-fizyczne*, vol. 17 (1906), p. 120.

p^{m-1} is the only group, besides the quaternion, which is completely defined by the facts that it is non-abelian and that it is generated by two operators which transform each other into given powers. Any series of operators such that each is transformed into a power of itself by all the others generates an abelian or a metabelian group which is therefore the direct product of its Sylow subgroups, but not every metabelian group can be generated in this way.*

If three operators are such that the product of any two is equal to the third, they generate either a dihedral group or the quaternion group.† If n different operators are commutative and if each of them is the product of all the others, they generate the direct product of a cyclic group whose order divides $2(n-2)$ and an abelian group of order 2^{α} and of type $(1, 1, 1, \cdots)$. Moreover, any such direct product may be generated by n operators which satisfy the given conditions. If the order of one of these operators is divisible by 4, all of them are of the same order. Every possible symmetric group whose degree exceeds 3 may be generated by five non-commutative operators such that each of them is the continued product of the other four, arranged in a given order. The groups generated by four operators such that each is the product of the other three, arranged in a given order, are similar to the dihedral type and have been completely determined.‡

It is well known that the groups of genus zero exhibit the most interesting generational relations. All of these have recently been generalized by replacing two of the three ordinary equations which define such a group by a single one while the third is maintained in the usual form. A large number of new and comparatively simple definitions of groups are thus obtained.§ Closely related to the groups of genus zero are those which may be defined by the orders of two operators and the order of their commutator. When two operators of order 2 have a commutator of an even order, they generate a group whose order is four times the order of this commutator; when they have a commutator of odd order, they generate a group whose order is either twice or four times the order of their commutator. If an operator of order 2 and an operator of order 3 have a commutator of order 2, they generate either the tetrahedral group or the direct product of this group and the group of order 2. If two operators of order 3 have a commutator of order 2, they generate a group whose order is one of the four numbers 12, 36, 144, 288.[*]

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* *Quar. Jour. of Mathematics*, vol. 37 (1906), p. 286.

† BULLETIN, vol. 13 (1907), p. 382.

‡ BULLETIN, vol. 13 (1907), p. 433.

§ *Transactions Amer. Math. Society*, vol. 8 (1907), p. 1.

Netto recently published a very elementary article on the construction of abstract groups which are generated by two operators,[†] in which he obtains a number of well-known results by the use of the most elementary principles and also arrives at a few interesting new relations. It is to be regretted that he failed to indicate where his work has contact with older publications. For instance, the conditions $a^3 = 1$, $b^3 = 1$, $ba = a^2b^2$ on page 255 of Netto's article are clearly equivalent to the well known abstract definition of the tetrahedral group, since $(ba)^2 = 1$, ba being equal to its inverse; yet nothing whatever is stated about this and the reader who is not familiar with the subject would naturally infer that the results were due to Netto. The last section of the article is devoted to the proof that the group generated by two operators of order 4 whose product is of order 2 has an infinite number of distinct operators. This is a very special case of an older general theorem in regard to the groups that may be generated by two operators when only their orders and the order of their product are given.[‡]

In volume 9 of the BULLETIN, Dickson gave useful generational relations for the simple groups of orders 360, 504, and 660, and in volume 35 of the *Proceedings of the London Mathematical Society* he gave such relations for an abstract simple group of order 4080 as well as for the more general abstract group which is simply isomorphic with the linear fractional group in $GF[2^n]$. In a later volume of the latter journal Bussey gave the generational relations for an abstract group simply isomorphic with $LF[2p^n]$.[§] Le Vavasseur, Potron, and Neikirk have given a very large number of generational relations as regards the groups of order p^m.[‖]

One of the most fundamental recent theorems relating to abstract groups is due to Frobenius. It affirms that the number of the operators whose nth power is equal to a given operator A is divisible by the highest common factor of n and g, where g represents the number of the operators of the group which are commutative with A.* Another recent fundamental theorem relates to the form of the total number of cyclic groups of order p^m $(m > 1)$ and asserts that the number of cyclic subgroups of order p^m in any group G is always of the form kp whenever the Sylow subgroups of order p^a in G are non-cyclic, and that the number of subgroups of order p in such a G is of the form $1 + p + kp^2$, p being an odd prime number.† If a group of order 2^m contains an odd number of cyclic subgroups of order 2^a $(\alpha > 2)$ this number must be unity and the group of order 2^m contains a cyclic subgroup of order 2^{m-1}. Moreover the number of operators of order 2 in any group of order 2^m which does not involve an operator of order 2^{m-1} is always of the form $4k + 3$, k being a positive integer. If a group of order 2^m contains exactly two cyclic subgroups of order 2^{m-1}, the number of the operators of order 2 is also of the form $4k + 3$; but when it contains only one such cyclic subgroup, this number is of the form $4k + 1$.‡ If a Sylow subgroup of order p^a in G is cyclic, the number of the subgroups of order p^m is evidently of the form $1 + kp$.

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[*] BULLETIN, vol. 13 (1907), p. 371. Cf. *Proc. Amer. Philosophical Society*, vol. 46 (1907), p. 146.

[†] Netto, *Crelle*, 128 (1905), p. 243.

[‡] *Amer. Jour. of Mathematics*, vol. 24 (1902), p. 96.

[§] Bussey, *Proc. London Math. Society*, vol. 3 (1905), p. 296.

[‖] Le Vavasseur, loc. cit., p. 26; Potron, Thèse, Paris, 1904; Neikirk, thesis, *Publications of the University of Pennsylvania.*

The study of the groups whose subgroups possess given properties has led to interesting results. From the properties of the non-abelian groups in which every subgroup is abelian, Dickson deduced the form of the order of groups which are necessarily abelian.§ This result is closely related to properties of the orders of groups which are necessarily cyclic. These properties were stated in my first report, page 235. The results as regards the non-abelian groups in which every subgroup is abelian have been extended by the determination of all the non-abelian groups in which every subgroup is either abelian or hamiltonian, as well as all those in which every subgroup is either abelian or dihedral.‖ These conditions led to remarkable elementary categories of groups which will probably continue to be useful, in view of their simple definitions.

All of the groups which were considered in the preceding paragraph are solvable. This is also true of the category which satisfy the condition that any two conjugate operators are commutative, as well as of the historically more interesting category composed of the groups whose orders are of the form $p^\alpha q^\beta$, p and q being prime numbers.* Special cases of such groups had been proved solvable by Frobenius, Jordan, Cole, and others before Burnside succeeded in establishing this property for the entire category. This seems the most interesting result as regards solvability which was reached during the period under consideration.

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* Frobenius, *Berliner Sitzungsberichte*, 1903, p. 987.

† *Proc. London Math. Society*, vol. 2 (1904), p. 142.

‡ *Transactions Amer. Math. Society*, vol. 6 (1905), p. 58.

§ Dickson, *Transactions Amer. Math. Society*, vol. 6 (1905), p. 201.

‖ *Transactions Amer. Math. Society*, vol. 8 (1907), p. 1; *Amer. Jour. of Mathematics*, vol. 29 (1907), p. 289.

Bachmann has observed that the theory of congruences and Fermat's theorem are based upon the "fundamental mathematical concept of groups" and he employed the elements of this subject in his classic work on number theory. A large number of relations between number theory and group theory have recently been considered.† In many instances the known theorems of these subjects express the same fundamental facts in different languages. This is the case as regards the generalized Wilson's theorem and the theorem giving the continued product of all the operators of an abelian group,‡ which was first stated by Zsigmondy. As Zsigmondy's definition of a group is not in accord with later usage, some of the results cannot be directly employed and hence his valuable memoir has not received the notice which it deserves. For instance, it gives a correct statement of the possible types of subgroups of abelian groups,§ while Burnside in his later Theory of Groups came to erroneous conclusions, as has been pointed out once more by Hilton in his recent article in the *Proceedings of the London Mathematical Society*, volume 5 (1907), page 1. A number of other known results are proved in this article without giving due references.

If a group contains one invariant subgroup of index p, the number of the invariant subgroups of this index is of the form $1 + p + p^2 + \cdots + p^\lambda$, p being any prime number. In particular, the number of subgroups of half the order of any group is some number of the series 0, 1, 3, 7, 15, $\cdots$. The value of λ in the preceding expression cannot exceed $m - 1$ when p^m is the order of a Sylow subgroup of a group. If $p = 2$ and

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* Burnside, *Proc. London Math. Society*, vol. 1 (1904), p. 388; also, vol. 35 (1903), p. 28.

† *Amer. Jour. of Mathematics*, vol. 27 (1905), p. 315.

‡ Fite, *Annals of Mathematics*, vol. 6 (1904), p. 7.

§ Zsigmondy, *Monatshefte für Math. und Physik*, vol. 7, (1896) p. 205

$m = 1$, λ must have this maximal value. The possible groups of order p^m when $\lambda = m - 2$ have been considered under the heading "Generalization of the hamiltonian groups" and the analogies between these groups and the hamiltonian groups have been studied. For smaller values of λ the difficulty of the problem of determining all the possible groups greatly increases, so that the case when $\lambda = m - 3$ has not yet been completely solved.* Another generalization of the hamiltonian groups is due to Wendt and was published in the *Mathematische Annalen* almost simultaneously with the appearance of a paper dealing with the same question, in a somewhat more restricted manner, and bearing the title "The groups in which every subgroup of composite order is invariant." † The number of the possible invariant subgroups of index p^2 has also been studied but the results are less simple than those in regard to invariant subgroups of index p.‡ Several interesting general theorems have been deduced from the one which affirms that the number of cyclic subgroups of order p^m in any group whose Sylow subgroups of order p^a are non-cyclic is divisible by p whenever $p > 2$ and $m > 1$. One of these relates to the number of cyclic subgroups of any order $k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda}$, $p_1 < p_2 < \cdots < p_\lambda$; $p_1, p_2, \cdots, p_\lambda$ being distinct prime numbers. If any group contains more than one cyclic subgroup of order k, it contains at least p_1 such subgroups, and if it contains exactly p_1 such subgroups, then $\alpha_1 > 1$ and it contains only one subgroup of order $p_\lambda^{\alpha_\lambda}$ $(\lambda > 1)$. In particular, there is no group which contains exactly two cyclic subgroups of the same order when this order is either odd or twice an odd number while there is an infinite number of different groups which contain exactly two cyclic subgroups whose order is an arbitrary number divisible by 4. A necessary and sufficient condition that a group is the direct product of its Sylow subgroups is that the nth power of each of its operators is contained in every subgroup of index n, for every possible value of n.§

Characteristic subgroups have the same property with respect to transformations under the group of isomorphisms as invariant subgroups have as regards transformations under the group. As the group of isomorphisms is continually receiving more attention, so the characteristic subgroups are continually playing a more prominent rôle. The groups in which every subgroup is characteristic are cyclic, but all others with the exception of the elementary groups, that is, those which are either simple or the direct product of simply isomorphic simple groups, have both characteristic and non-characteristic subgroups. The most important characteristic subgroups are generated by the commutators of the groups which are not perfect. All the characteristic subgroups of an abelian group have been determined and it has been proved that they have a common subgroup known as the fundamental characteristic subgroup. The subgroup generated by the ith commutators is also characteristic.*

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* *Transactions Amer. Math. Society*, vol. 6 (1905), p. 326.

† BULLETIN, vol. 12 (1906), p. 379; *Archiv der Math. und Physik*, vol. 11 (1906), p. 76.

‡ *Comptes rendus*, vol. 140 (1905), p. 32.

§ *Amer. Math. Monthly*, vol. 13 (1906), p. 10.

The characteristic subgroup generated by all the invariant operators of a group corresponds to the identity of the group of cogredient isomorphisms and hence it is of such unusual importance as to have received special names. In view of its dependence on this quotient group, it was first called the *cogredient subgroup*,† but in the text-book of de Séguier and in other French literature it has been denoted by the briefer term *the central*, while the corresponding quotient group has been called *the cogredient*. This quotient group is abelian in an important category of groups studied by Fite and others and known as metabelian groups.‡

The theorem that every group of order p^m contains an abelian subgroup of order p^a whenever $m > \frac{1}{2}a(a-1)$ has been extended by showing that the number of these abelian subgroups is of the form $1+kp$ and hence at least one of them is invariant, and also by proving that every group of order 64 contains an abelian subgroup of order 16, while there are groups of order p^6 for all odd values of p which do not contain an abelian subgroup of order p^4.§ In the Decennial Publications of the University of Chicago Moore determines all the subgroups of the generalized finite modular group. As noted in the preceding report, Wiman investigated the same problem, but Moore had practically completed his investigations before the appearance of Wiman's memoir.‖ The groups in which every two conjugate operators are commutative and those in

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*Fite, *Transactions Amer. Math. Society*, vol. 7 (1906), p. 61.

†*Amer. Math. Monthly*, vol. 5 (1898), p. 221.

‡Fite, *Transactions Amer. Math. Society*, vol. 7 (1902), p. 331.

§*Messenger of Mathematics*, vol. 36 (1907), p. 188.

‖Cf. *Jahrbuch über die Fortschritte der Mathematik*, vol. 34 (1905), p. 172.

which every pair of commutative operators, apart from identity, are conjugate have been investigated by Burnside and Rietz respectively.* It was found that the symmetric group of order 6 is the only group which satisfies the latter condition.

In volume 6 of the *Transactions* of this Society, Professors Moore, Dickson and Huntington extended their earlier results in regard to the definitions of an abstract group. Moore proves that one of the postulates in his earlier definition is redundant and establishes the mutual independence of the postulates in the new definition. Huntington gives a brief historical sketch of abstract definitions of groups and fields, and states that the first abstract definitions of fields are due to Dickson and himself, and are natural extensions of the sets of independent postulates which had already been given for groups. In various memoirs Dickson has developed a theory of groups in an arbitrary field defined by finite groups and has shown, in particular, that when the field is the general Galois field of order p^n there is a doubly infinite system of finite groups which corresponds to each given finite group.† The work has close contact with the two earlier papers of Burnside, "On the continuous group that is defined by any given group of finite order."

If the number of operators of order 2 in an abelian group is at least equal to half the order of the group, then the group contains no operator whose order exceeds 2. There are, however, many different types of non-abelian groups in which the number of operators of order 2 is more than half the order of the group. The number of operators whose orders exceed 2 in such a group cannot be less than one fourth of the order of the group, and if it exceeds this number it must be at least equal to one third of the order of the group. For the groups of order 2^n all the possible ratios between the number of the operators whose orders exceed 2 and the order of the group have been determined on the hypothesis that this ratio is less than one half, and the maximum number of operators of order 2 in a group of any given order has been found.‡

In the enumeration of all the possible groups which satisfy given conditions encouraging progress has been made. As noted in my preceding report, the groups whose orders are the products of four prime numbers had all been considered with the exception of the case when the order is of the form p^2qr. The possible groups whose orders are of this form have recently been enumerated by Glenn.* The most extensive enumeration which has appeared during the period covered by the present report is that given in Potron's Paris University thesis. The author of this work aims to give a complete enumeration of all the possible groups of order p^6 together with some general types of groups of order p^m. Among the latter are those in which every subgroup of order p^{m-2} is abelian while the group itself is metabelian. In a later note the author removes the latter restriction and makes some corrections of his enumeration of the groups of order p^6.† He uses the term *commutant* instead of the older terms derivative, or commutator subgroup.

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* Rietz, *Transactions Amer. Math. Society*, vol. 5 (1904), p. 500.

† Dickson, University of Chicago Decennial Publications, vol. 9 (1902), p. 35; *Transactions Amer. Math. Society*, vol. 3 (1902), p. 285.

‡ BULLETIN, vol. 13 (1907), p. 235.

Another extensive enumeration is due to Neikirk and is devoted to the group of order p^m $(p > 2)$ which involve cyclic subgroups of order p^{m-3}.‡ The special case where $p = 2$ has been considered more recently by Miss McKelden.§ Among the less extensive enumerations which are not mentioned in connection with other subjects in the present report are the following: The groups which contain less than 15 operators of order 2 and are generated by these operators; ‖ the groups of order p^m which contain exactly p cyclic subgroups of order p^a, those of order 2^m which contain an odd number of cyclic subgroups of composite order, and those which contain only three operators which are squares of other operators in the group.¶

All the operators of a group of composite order are contained in its cyclic subgroups but it is not always possible to select these subgroups in such a way that every two have only identity in common. The more general question of finding the properties of a group which satisfies the condition that all its operators are contained in a series of subgroups such that any two have only identity in common has been partially studied. The most interesting theorems that have been established may be stated as follows: If the group is non-abelian and of order p^m, then only one of these subgroups can involve operators of order p^2. If it is abelian, its order must be of the form p^m and its type must be $(1, 1, 1, \cdots)$.*

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* Glenn, *Transactions Amer. Math. Society*, vol. 7 (1906), p. 137.

† Potron, *Bulletin de la Société math. de France*, vol. 32 (1904), pp. 296, 300.

‡ Neikirk, *Publications of the University of Pennsylvania*, No. 3, 1905.

§ *Amer. Math. Monthly*, vol. 13 (1906), p. 121.

‖ *Amer. Jour. of Mathematics*, vol. 29 (1907), p. 1.

¶ *Transactions Amer. Math. Society*, vol. 7 (1906), pp. 94 and 238; vol. 6 (1905), p. 58.

Dickson has found a new system of simple groups by the study for modulus 2 of a linear group on 7 variables which he had investigated before for fields not having this modulus. The problem for modulus 2 required a different analysis and led to a simple group of order $2^{6q}(2^{6q}-1)(2^{2q}-1)$, $q > 1$. For $q = 1$ the group has a simple subgroup of index 2 and of order 6048.† The following theorem is frequently useful in the study of simple groups, especially if they are represented as substitution groups. If p^a is the highest power of the prime p which divides the order of a simple group K and if K contains less than $(p+1)^2$ subgroups of order p^a, then each of these Sylow subgroups is transformed into itself by a maximal subgroup of K.‡

In view of the fundamental importance of Sylow's theorem we add the following new statement of it, even though this statement can be readily derived from the one usually given. If the order of a group G is $p^a m$, p being any prime which does not divide m, G contains at least one subgroup P_a of order p^a. Moreover if P_a contains only one subgroup P_β of a given type, then all the subgroups of this type are conjugate under G and their number is of the form $1 + kp$. The order of G may be written in the form $p^\beta n(1 + kp)$, $p^\beta n$ being the order of the largest subgroup which transforms P_β into itself.§ By letting $\beta = \alpha$ we obtain an ordinary form of this theorem.

(*To be continued.*)

NOTES.

THE concluding (October) number of volume 29 of the *American Journal of Mathematics* contains the following papers: "On twisted quintic curves," by E. C. COLPITTS; "Attraction of the homogeneous spherical segment," by G. W. HILL; "On a certain class of algebraic translation surfaces," by J. EIESLAND; "Group characters of various types of linear groups," by H. E. JORDAN.

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* BULLETIN, vol. 12 (1906), p. 446.

† Dickson, *Mathematische Annalen*, vol. 60 (1905), p. 137.

‡ *Comptes rendus*, vol. 136 (1903), p. 294.

§ *Annals of Mathematics*, vol. 5 (1904), p. 187.

THE fifty-seventh annual meeting of the American association for the advancement of science will be held at Chicago, December 30 to January 4, 1908, under the presidency of Professor E. L. NICHOLS, of Cornell University. Professor E. O. LOVETT is vice-president and chairman of section A (mathematics and astronomy) and Professor L. G. WELD is secretary. The address before Section A by the retiring vice-president, Professor EDWARD KASNER, will be on "Geometry and mechanics."

THE royal academy of sciences of Madrid announces the following prize problem for the year 1908:

"A concise exposition of the fundamental principles of nomography which are strictly necessary for the composition of a system of abaci or nomograms, as yet unknown and applicable with manifest advantage over any other procedure to the resolution of a series of questions, interesting in theory and useful in practice, referring to the physical and mathematical sciences."

The prize will be awarded in three classes. The first class, or prize proper, consists of a diploma of award, a gold medal, a money prize of 1500 pesetas, and a hundred reprints of the successful memoir for the author. The second prize consists of a diploma, a gold medal, and a hundred reprints. The third prize is a diploma. Competing memoirs must be written in Spanish or Latin and sent to the Secretaria de la Academia, 36 calle de Valverde, Madrid, before December 31, 1908.

THE late Dr. P. WOLFSKEHL, of Darmstadt, left the sum of 100,000 marks to the academy of sciences of Göttingen, to be awarded to the person who first presents a rigorous proof of the celebrated Fermat theorem: "The equation $x^n + y^n = z^n$ $(n > 2)$ can never be satisfied by integers." Until this end has been attained, the interest on this sum is to be applied to mathematical purposes at the discretion of the academy.

THE time for submitting papers in competition for the Guccia prize (see BULLETIN, volume 11, page 167) expired July first. Three memoirs were received, bearing the titles: "Sur les courbes gauches de direction" (234 pages); "Sur quelques propriétés arithmétiques des courbes algébriques planes ou gauches" (16 pages); "Grundlage zu einer Bewegungstheorie des Kreises und der Kugel" (45 pages). On the sixth of July another memoir was received, bearing the title "Ueber die Uniformisierung beliebiger algebraischer Kurven" (70 pages), but under the rules of the competition it could not be accepted. The result of the competition is to be announced at one of the sessions of the fourth international congress of mathematicians at Rome next April.

THE Hamburg mathematical society, founded in 1690, chiefly by the efforts of its first president, Heinrich Meissner, consisted at the time of its organization of six resident and nine non-resident members, and assumed the title "Kunstrechnungsübende Societät." That the society had a higher purpose than the solution of riddles is shown by the fact that it has continued in activity over two hundred years; it is the oldest mathematical society extant, and the oldest scientific society of Germany, except the Leopold-Carolus academy of Halle, which was founded in 1652. Its publications consisted at first of an annual letter or report and various subsidized books written by prominent members, the most important of which was the Mathematisches Sinnenconfect, written by P. Halcke in 1719. In the latter half of the eighteenth century over half the members resided in Holland; owing to the inconvenience of attending the monthly meetings at Hamburg, these members were largely influential in organizing the mathematical society of Amsterdam in 1778. From the time of the French revolution until the Franco-Prussian war, the chief activity of the society was the cultivation of applied mathematics, and most of its members were engineers, but since 1870 more interest has been shown in the broader view of mathematics. In 1873 an autographed abstract of the papers read before the society was appended to the yearly report and sent to all members. This proved so useful that it was continued until 1881, at which time the first number of the *Mitteilungen* was issued. The society now has over 60 members and possesses a substantial library. Although its influence has not been widely extended, yet the Hamburg mathematical society has been a most potent factor in maintaining a high standard of excellence in the schools of the city.

THE *Annuario* of the Circolo mathematico di Palermo issued in September, contains in addition to the usual alphabetic list of its members, a chronological list containing both former and present members distinguished by different type, a list of all the other learned societies to which any members belong, and the names of all such members in each, a similar list of universities and other institutions, and finally, a list of periodicals whose editorial staff includes one or more members of the Circolo. The society has 511 members, of whom 81 are Americans. During the year 1906 the membership increased by 107.

The Circolo has undertaken to publish the mathematical works and scientific correspondence of Paolo Ruffini, including the Theoria generale delle equazioni, which appeared at Bologna in 1799. The necessary funds have been generously provided by Professor G. B. GUCCIA, director of the *Rendiconti*.

The following university courses in mathematics are announced:

CAMBRIDGE UNIVERSITY. *Michaelmas term*, beginning October 14. — By Professor A. R. FORSYTH: Differential geometry, three hours. — By Professor G. H. DARWIN: Dynamical astronomy, three hours. — By Professor Sir R. S. BALL: Planetary theory, three hours. — By Professor J. LARMOR: Electricity and magnetism, three hours. — By Dr. E. W. HOBSON: Spherical harmonics and allied functions, three hours. — By Dr. H. F. BAKER: Introduction to the theory of functions, three hours, Theory of groups, three hours. — By Mr. B. A. HERMAN: Hydrodynamics, three hours. — By Mr. H. W. Richmond: Analytic geometry, three hours. — By Dr. A. N. WHITEHEAD: Principles of mathematics, three hours; Non-euclidean geometry, three hours. — By Mr E. W. BARNES: Linear differential equations, three hours.— By Mr. A. MUNRO: Hydrodynamics and sound, three hours. — By Mr. J. H. GRACE: Invariants and geometric applications, three hours. — *Lent term*, beginning January 16, 1908. — By Professor A. R. FORSYTH: Differential geometry, three hours. — By Professor G. H. DARWIN: Figure of the earth, three hours. — By Professor J. LARMOR: Electrodynamics with optical applications, three hours. — By Dr. E. W. HOBSON: Differential equations and expansions of mathematical physics, three hours. — By Dr. H. F. BAKER: Solid geometry, three hours; Theory of functions, three hours. — By Mr. B. A. HERMAN: Hydrodynamics, three hours.— By Mr. H. W. RICHMOND: Analytic geometry, three hours. — By Dr. A. N. WHITEHEAD: Principles of mathematics, three hours; Non-euclidean geometry, three hours. — By Mr. E. W. BARNES: Hypergeometric series, three hours. — By Mr. A. BERRY: Elliptic functions, Bessel functions and Fourier series, three hours.—By Mr. C. T. BENNETT: Line geometry, three hours.—By Mr. A. YOUNG: Theory of invariants, three hours; Discontinuous groups, three hours. *Easter term*, beginning April 27, 1908.—By Professor A. R. FORSYTH: Differential geometry, three hours.—By Professor J. LARMOR: Theory of gases, three hours.—By Dr. H. F. BAKER: Theory of functions, three hours.—By Mr. B. A. Herman: Hydrodynamics, three hours.—By Mr. H. W. RICHMOND: Projective geometry, three hours.—By Dr. A. N. WHITEHEAD: Non-euclidean geometry, three hours.—By Mr. A. BERRY: Elliptic functions, three hours.—By Mr. A. MUNRO: Line geometry, three hours.—By Mr. G. H. HARDY: Integral functions, three hours.

UNIVERSITY OF VIENNA. *Winter Semester*.—By Professor G. v. ESCHERICH: Introduction to the theory of functions, five hours; Seminar, three hours.—By Professor F. MERTENS: Algebra, five hours; Seminar, three hours.—By Professor W. WIRTINGER: Calculus, five hours; Seminar, three hours.—By Professor G. KOHN: Analytic geometry, four hours; Differential geometry, two hours.—By Professor E. BLASCHKE: Introduction to the mathematics of statistics, three hours.—By Professor K. CARDA: Infinite groups, two hours.—By Dr. J. PLEMELJ: Theory of numbers, three hours.—By Dr. H. HAHN: Foundations of geometry, two hours.—By Dr. L. HANNI: Theoretic arithmetic, two hours—By Dr. L. SCHRUTKA: Finite discrete groups, two hours.

THE October number (volume 16, number 9) of the *Jahresbericht der deutschen Mathematiker-Vereinigung* contains an interesting abstract of the scheme of old age and disability pensions of full professors in the German universities and technical schools.

THE works of the Japanese mathematician SEKI, a contemporary of Newton, entitled Shichibusho (Seven Books), are soon to appear in one volume under the auspices of the Tokio Mathematico-Physical Society, in commemoration of the two hundredth anniversary of the death of this noted scholar.

BARON D. KIKUCHI, formerly professor of mathematics in the Imperial University of Tokio, returned to that city on September 14 after a trip around the world during which he gave a number of addresses in England. The baron is chairman of a committee which is about to undertake the work of compiling a history of native Japanese algebra.

THE following have been elected honorary members of the London mathematical society: Professor G. CASTELNUOVO, of the University of Rome, Dr. G. W. HILL, of New York, Professor C. JORDAN, of the College of France, Professor V. VOLTERRA, of the University of Rome.

THE following mathematicians have been decorated by the German emperor: Professor D. HILBERT, of the University of Göttingen, with the order of the crown of the third class; Professor W. KIEPERT, of the technical school of Hanover, with the order of the crown of the second class; Professor R. v. LILIENTHAL, of the University of Münster, with the order of the red eagle of the fourth class; Professor W. KILLING, of the University of Münster, with the order of the crown of the third class.

PROFESSOR A. R. FORSYTH, of the University of Cambridge, received the honorary degree of doctor of laws at the 25-year jubilee of the University of Liverpool.

DR. M. ABRAHAM, of the University of Göttingen, has been promoted to an associate professorship of mathematics.

DR. G. HERGLOTZ, of the University of Göttingen, has declined the call to the University of Freiburg, Switzerland.

DR. M. FRÉCHET, of the lycée at Besançon, has been appointed professor of special mathematics at the lycée at Nantes.

DR. G. NITALI has been appointed docent in mathematics at the University of Genoa.

PROFESSOR G. LAURICELLA, of the University of Catania, has been elected a corresponding member of the Accademia dei Lincei at Rome.

PROFESSOR D. A. MURRAY, of Dalhousie College, has been appointed to the chair of applied mathematics at McGill University, as successor to the late Professor G. H. Chandler.

MR. MURRAY MACNEILL, assistant professor of mathematics at McGill University, has been appointed professor of mathematics at Dalhousie College.

PROFESSOR E. H. MOORE, of the University of Chicago, will spend most of the present academic year in Italy, leaving New York about New Year's and returning in October.

PROFESSOR A. N. SKINNER, of the U. S. Naval Observatory, has retired under the age limit from his position as professor of mathematics. MR. H. L. RICE has been appointed his successor.

AT Brown University, Professor N. F. DAVIS has been granted a year's leave of absence. DR. R. G. D. RICHARDSON, recently of Yale University, has been appointed assistant professor of mathematics, and DR. H. H. CONOVER, also recently of Yale, has been appointed instructor in mathematics.

MR. FLOYD FIELD has been appointed junior professor of mathematics at the Georgia School of Technology.

DR. B. L. NEWKIRK, of the Lick Observatory, has been appointed assistant professor of mathematics and mechanics in the University of Minnesota.

AT Iowa College, Grinnell, Ia., Dr. R. B. MCCLENON has been promoted to an assistant professorship of mathematics; Mr. R. E. HAWLEY has been appointed instructor in mathematics.

MR. J. D. SUTER and MR. E. JORDAN have been appointed instructors in mathematics at Stanford University.

The following academic appointments are also announced: Mr. J. B. REYNOLDS, instructor in mathematics at Lehigh University; Mr. G. R. CLEMENTS, instructor in mathematics at Williams College; Mr. J. W. MITCHELL, instructor in mathematics at the Agricultural College of Texas; Mr. R. B. STONE, instructor in mathematics at Bowdoin College; Mr. — BREWSTER and Mr. — COOK, tutors in mathematics at the College of the City of New York; Miss M. E. WELLS has resumed her instructorship at Mount Holyoke College.

Catalogues of second-hand mathematical books: G. E. Stechert and Co., 129 West 20th Street, New York, catalogue 18, exact sciences, including journals, 98 pages. — A. Hermann, 6 rue de la Sorbonne, Paris, catalogue 89, journals, mathematics, astronomy and geodesy, 2064 titles; catalogue of theses, 100 entries in mathematics and astronomy.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Achitsch (A.). Ein neues Integrationsverfahren. (Progr.) Pola, 1906. 8vo. 52 pp.

Adhémar (R. d'). Les équations aux dérivées partielles à caractéristiques réelles. Paris, Gauthier-Villars, 1907. 8vo. 86 pp. Fr. 2.00

Andriessen (C.). Ueber Erzeugniss kongruenter Grundgebilde. (Diss.) Strassburg, 1906. 8vo. 45 pp.

Apfelstedt (M.). Ueber eine Gattung von projektiven Transformationsgruppen in fünf Veränderlichen. (Diss.) Greifswald, 1906. 8vo. 75 pp.

Bachelet (C.). Cenni elementari sui limiti e sui numeri incommensurabili, ad uso delle scuole tecniche e normali. Torino, Cossone, 1907. 8vo. 22 pp.

Barbarin (P.). La géométrie non-euclidienne. 2^e^ édition. Paris, Gauthier-Villars, 1907. 8vo. 92 pp. Fr. 2.00

Büchel (H.). Ueber ein nicht holonomes System: Die Rollbewegung einer Kugel in einer Kugelschale. (Diss.) Strassburg, 1906. 8vo. 40 pp.

Burkhardt (H.). Vorlesungen über die Elemente der Differential- und Integralrechnung und ihre Anwendung zur Beschreibung von Naturerscheinungen. Leipzig, Teubner, 1907. 8vo. 11 + 252 pp. Cloth. M. 6.00

Cantor (M.). Vorlesungen über Geschichte der Mathematik. Vol. IV. Von 1759 bis 1799. 2te Lieferung. Abschnitt XXI von E. Netto: Kombinatorik, Wahrscheinlichkeitsrechnung, Reihen, Imaginäres. Abschnitt XXII von V. Bobynin: Elementare Geometrie. Leipzig, Teubner, 1907. 8vo. Pp. 201–402. M. 5.60

Cappilleri (A.). Einführung in die Ausgleichungsrechnung. (Methode der kleinsten Quadrate.) Wien, 1907. 8vo. 138 pp. M. 4.00

Carosi (E.). Sviluppo di funzioni in serie doppie trigonometriche. Foligno, Campi, 1907. 8vo. 79 pp.

Cornacchia (G.). Sulla congruenza $x^n - y^n = x^n$ mod. p. Tempio, Tortu, 1907. 8vo. 18 pp.

Dannacher (S.). Zur Theorie der Funktionen des elliptischen Zylinders. Frauenfeld, 1906. 4to. 38 pp. M. 2.40

Del Re (A.). Lezioni di algebra della logica, dettate nella r. università di Napoli. Napoli, r. Accademia delle scienze fisiche e matematiche, 1907. 8vo. 104 pp. L. 3.50

Doria (G. A.). Studio sulle funzioni intere semplici. Bologna, Cuppini, 1907. 8vo. 30 pp.

Durhold (P.). Ueber einen Kreisbündel sechster Ordnung. (Diss.) Jena, 1907. 8vo. 57 pp.

Encyclopédie des sciences mathématiques pures et appliquées. Publiée sous les auspices des académies des sciences de Göttingue, de Leipzig, de Munich et de Vienne avec la collaboration de nombreux savants. Edition française. Rédigée et publiée d'après l'édition allemande sous la direction de J. Molk. Tome I (1er volume): Arithmétique. Rédigé dans l'édition allemande sous la direction de F. Meyer. Fascicule 2 (pp. 161–328). Leipzig, Teubner, 1907. 8vo. M. 4.20

Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band III: Geometrie, redigiert von F. Meyer. Teil 1, Heft 1: F. Enriques, Prinzipien der Geometrie; H. von Mangoldt, Die Begriffe "Linie" und "Fläche;" M. Dehn und P. Heegaard, Analysis Situs. 220 pp. Band IV: Mechanik, redigiert von F. Klein und C. H. Müller. Teil 2, II, Heft 1: C. H. Müller und A. Timpe, Die Grundgleichungen der mathematischen Elastizitätstheorie; O. Tedone, Allgemeine Theorie der mathematischen Elastizitätslehre. Leipzig, Teubner, 1907. 8vo. 124 pp.

Enders (M. A.). Ueber die Darstellung der Raumkurve vierter Ordnung vom Geschlecht 1 durch Thetafunktionen. (Diss.) Strassburg, 1907. 4to. 29 pp.

Fouët (E. A.). Leçons élémentaires sur la théorie des fonctions analytiques. 2e édition entièrement refondue. 1er Tome: Les fonctions en général. Paris, Gauthier-Villars, 1907. 8vo. 13 + 113 pp. Fr. 3.50.

Franz (K.). Ueber Kreisschnitte auf Flächen zweiter Ordnung, speziell über Kreisschnitte auf dem Ellipsoid. (Progr.) Hamburg, 1907. 4to. 23 pp.

Frobenius (G.). Ueber einen Fundamentalsatz der Gruppentheorie. II. Berlin, 1907. 8vo. 10 pp.

Geszner (E.). Ueber die Asymptotenkurven einer Schar Konoidflächen im allgemeinen und die des Cylindroids im besonderen. (Diss.) Münster, 1906. 8vo. 38 pp.

Graf (J. H.). Der Basler Mathematiker Leonhard Euler. Bern, 1907. 8vo. 24 pp. M. 0.80

Hauser (W.). Ueber Resultanten- und Discriminantenbildung in der Theorie der elliptischen Thetafunktionen. (Diss.) Erlangen, 1907. 4to. 40 pp.

Hemardinquer (C.). Notions de mathématiques supérieures (Calcul intégral et différentiel); licence ès sciences physiques, écoles techniques, constructeurs et praticiens. Avec une préface de C. A. Laisant. Paris, Paulin, 1907. 18mo. 6 + 144 pp. Fr. 3 00

Hobson (E. W.). The theory of functions of a real variable and the theory of Fourier's series. Cambridge, University Press, 1907. 8vo. 788 pp. 21s.

Huebner. See Reim.

Kerl (O.). Voranschläge der Genauigkeit beim trigonometrischen Punkteinschalten. (Diss.) Berlin, 1907. 8vo. 59 pp.

Klein (F.). See Vorlesungen, also Encyklopädie.

Kohlrausch (F. L.). Einführung in die Differential- und Integralrechnung nebst Differentialgleichungen. Berlin, Springer, 1907. 8vo. 7 + 191 pp. Cloth. M. 6.80

Laisant (C. A.). See Hemardinquer (C.).

Langenkamp (O.). Ueber Saccheris Untersuchungen des Parallelenaxioms. (Diss.) Münster, 1907. 8vo. 30 pp.

Loney (S. L.). The elements of coordinate geometry for matriculation. London, Macmillan, 1907. 8vo. 182 pp. 3s. 6d.

Magener (A.). Anallagmatische Punktkoordinaten im Kugelgebüsch und ihre Anwendung auf die nichteuklidische Geometrie. (Diss.) Strassburg, 1907. 8vo. 56 pp.

Mangelsdorf (E.). Eine neue Abbildung des linearen Strahlencomplexes auf den Punktraum. (Diss.) Strassburg, 1906. 8vo. 32 pp.

Meyer (F.). See Encyclopédie, also Encyklopädie.

Minkowski (H.). See Vorlesungen.

Molk (J.). See Encyclopédie.

Montel (P.). Sur les suites infinies de fonctions. (Thèse.) Paris, Gauthier-Villars, 1907. 4to. 110 pp.

Müller (C. H.). See Encyklopädie.

Müller (H.) Einführung in die Differential- und Integralrechnung. Zum Gebrauch an höheren Schulen bearbeitet. (H. Müllers mathematisches Unterrichtswerk.) Leipzig, Teubner, 1907. 8vo. 5 + 38 pp. M. 1.50

Nernst (W.) und Schönflies (A.). Einführung in die mathematische Behandlung der Naturwissenschaften. Kurzgefasstes Lehrbuch der Differential- und Integralrechnung mit besonderer Berücksichtigung der Chemie. 5te Auflage. München, Oldenbourg, 1907. 8vo. 12 + 371 pp. M. 12.50

Pedrotti (L.). Introduzione al calcolo differenziale colla teorica dei massimi e minimi. Milano, Società ed. Sonzogno, 1907. 16mo. 62 pp. L. 0.15

Pincherle (S.). Algebra complementare. Parte II: Teoria delle equazioni. 2ª edizione. Milano, Hoepli, 1907. 16mo. 8 + 174 pp.

Rehfeld (C.). Ueber Klassenbildung und Klassenzahl in algebraischen Funktionenkörpern. (Diss.) Strassburg, 1906. 8vo. 52 pp.

Reichel (W.). Ueber trilineare alternierende Formen in sechs und sieben Veränderlichen und die durch sie definierten geometrischen Gebilde. (Diss.) Greifswald, 1907. 8vo. 59 pp.

Reim. Das regelmässige Dodekaeder und Ikosaeder in ihren wechselseitigen Beziehungen, nach Angaben von Dr. Huebner dargestellt. (Progr.) Schweidnitz, Heege, 1907. 8vo. 28 pp. M. 1.50.

Roelcke (O.). Ueber die Bäcklundsche Transformation der Flächen konstanter Krümmung. (Diss.) Greifswald, 1907. 8vo. 41 pp.

Röhrborn. Potential und Anziehung eines verlängerten Rotationsellipsoids für den Fall, dass die Dichtigkeit gleich $k_0 + k_1xy + k_2xz + k_3yz$ ist. (Progr.) Hirschberg, 1907. 4to. 28 pp.

Schimmack (R.). See Vorlesungen.

Schönflies (A.). See Nernst (W.).

Schülke (A.). Behandlung der Differential- und Integralrechnung im Unterricht (Progr.). Königsberg, 1907. 8vo. 30 pp.

TANNENBERG (W. DE). Cours de géométrie analytique. 3e fascicule: Courbes planes en général (Courbes définies par leurs équations paramétriques). Paris, Vuibert, 1907. 4to. Pp. 89 to 181.

TOLEDO (L. O.). Introducción al estudio de las funciones de variable compleja. Madrid, Fortanet, 1907. 215 pp. P. 5.00

VOLTERRA (V.). Leçons sur l'intégration des équations différentielles aux dérivées partielles. Upsala, 1906. 4to. 83 pp.

VORLESUNGEN, mathematische, an der Universität Göttingen. Leipzig, Teubner, 1907. 8vo.

I. 1. Klein, F. Vorträge über den mathematischen Unterricht an den höheren Schulen. Bearbeitet von R. Schimmack. 1er Teil: Von der Organisation des mathematischen Unterrichts. 9 + 236 pp. Cloth. M. 5.00

II. Minkowski, H. Diophantische Approximationen. Eine Einführung in die Zahlentheorie. 8vo. 8 + 236 pp. Cloth. M. 8.00

VOSS (A.) Ueber Krümmung und konforme Transformation. München, 1907. 8vo.

WEISS (E.). Anzahlbestimmungen für das Strahlennetz (lineare Kongruenz). Breslau, 1907. 8vo. 64 pp. M. 2.40

WÜNSCHMANN (K.). Ueber Berührungsbedingungen bei Integralkurven von Differentialgleichungen. Greifswald, 1907. 8vo. 35 pp. M. 1.60

II. ELEMENTARY MATHEMATICS.

BACHELET (C.). Problemi e teoremi di geometria euclidea. I. Torino, 1907. 8vo. 11 pp.

BAKER (W. M.) A key to algebraic geometry. London, Bell, 1907. 8vo. 7s. 6d.

BOTZHEIM (A. DE). La quadrature du cercle; essais géométriques. St. Blaise, 1906. 4to. 6 pp. Fr. 1.50

BOURLET (C.). Précis d'algèbre contenant 573 exercices et problèmes, rédigé conformément aux programmes du 31 mai 1902 et du 27 juillet 1905. (Classes de troisième B, seconde et première C et D.) 4e édition, revue et complétée. Paris, Hachette, 1907. 16mo. 443 pp. Fr. 2.50

BURRAN (C.). Tafeln der Funktionen Cosinus und Sinus mit den natürlichen sowohl reellen als rein imaginären Zahlen als Argument (Kreis- und Hyperbelfunktionen). Mit Einleitung in deutscher, französischer und englischer Sprache. Berlin, 1907. 8vo. 20 + 63 pp. Cloth. M. 4.00

CALVITTI (G.). Elementi di trigonometria piana e sferica ad uso delle scuole secondarie. Ascoli, Piceno, 1907. 8vo. 92 pp.

CELENTANO (A.). Nozioni di algebra elementare per le scuole tecniche. Napoli, Casella, 1907. 16mo. 50 pp. L. 1.00

COMSTOCK (C. E.). Elementary Algebra. Part I. Peoria, Ill., Comstock, 1907. 12mo. Cloth. $1.10

CRÜWELL (E. R.). Regeln der Mathematik. 3te, vermehrte Auflage. Lehrgang I: Planimetrie und Grundlehren der Stereometrie. Berlin, 1907. 8vo. 114 pp. M. 10.00

DICKMANN. See HEILERMANN, also KOPPE.

ENRIQUES (F.). Fragen der Elementargeometrie. Aufsätze von M. Amaldi, E. Baroni, E. Daniele, G. Vitali und anderen, gesammelt und zusammengestellt. Deutsche Ausgabe von H. Fleischer. (2 Teile.) Teil II: Die geometrischen Aufgaben, ihre Lösung und Lösbarkeit. Leipzig, 1907. 8vo. 12 + 348 pp. Cloth. M. 9.00.

FLEISCHER (H.). See ENRIQUES (F.).

GRAF (J. H.). Zur Geschichte der mathematischen Wissenschaften an der ehemaligen Akademie und der Hochschule Bern. Bern, 1906. 8vo. 19 pp.

GRÉVY (A.). Compléments de géométrie, à l'usage des élèves des classes de mathématiques A et B (Programme du 27 juillet 1905). Paris, Vuibert, 1907. 8vo. 477 pp. Fr. 2.00

HAASE (E.). See WILK (E.).

HALBGEBAUER (H.). See MOČNIK.

HARRIS (G.). Heuristic geometry for elementary schools. London, Meiklejohn, 1907. 8vo. 9d.

HAWKINS (C.). Elementary trigonometry. London, Dent, 1907. 8vo. 310 pp. 4s.
With answers, 4s. 6d.

HEILERMANN and DIEKMANN. Lehr- und Uebungsbuch für den Unterricht in der Algebra an den höheren Schulen. Neu bearbeitet von K. Knops. 1er Teil, 12te Auflage. Essen, Baedeker, 1907. 8vo. 8 + 236 pp. M. 2.25

HÖHM (F.). Geometrische Anschauungslehre für die I. bis IV. Klasse der Mädchen-Lyzeen. 2 Teile. Wien, Tempsky, 1907. 8vo. 95 pp. M. 1.80

HUNTINGTON (E. V.). Four-place tables of logarithms and trigonometric functions. Abridged edition. Cambridge, Harvard Coöperative Society, 1907. 8vo. 28 pp. $0.35

KNOPS (K.). See HEILERMANN, also KOPPE.

KOPPE und DIEKMANN. Geometrie zum Gebrauche an höheren Unterrichtsanstalten. Ausgabe für Reallehranstalten, 3er Teil, 3te Auflage, bearbeitet von K. Knops. Essen, Baedeker, 1907. 8vo. 7 + 288 pp. M. 3.20

LARA (F.). Descripción y uso de las tablas trigonométricas de Schrön, con varios ejercicios resueltos. Toledo, Gomez Ulenor, 1907. 57 pp. P. 1.50

LENNES (N. J.). See SLAUGHT (H. E.).

MATRICULATION mathematics papers: Being the papers in elementary mathematics set at the matriculation examination of the University of London from Jan. 1897 to Jan. 1907. London, Clive, 1907. 8vo. 124 pp. 1s. 6d.

MATRICULATION model answers: Mathematics, Sept. 1904 to June 1907. London, Clive, 1907. 8vo. 130 pp. 2s.

MOČNIK. Geometrie und geometrisches Zeichnen für Knaben-Bürgerschulen. Bearbeitet von H. Halbgebauer. Ausgegeben in 1 Bande. Wien, Tempsky, 1907. 8vo. 216 pp. M. 2.50

MODEL answers to intermediate pure mathematics. New edition. London, Clive, 1907. 8vo. 154 pp. 2s. 6d.

MOSBACHER (L.). Die Definitionen und Regeln der elementaren Algebra und ihre Anwendungen. Nürnberg, 1907. 8vo. 43 pp. M. 1.60

MÜLLER (H.). Die Mathematik auf den Gymnasien und Realschulen. Für den Unterricht dargestellt. 1er Teil: Die Unterstufe. (Lehraufgabe der Klassen Quarta bis Unter-Sekunda). Ausgabe B. Für reale Anstalten und Reformschulen. Leipzig, Teubner, 1907. 8vo. 8 + 200 pp. M. 2.20

OTTO (F.) und SIEMON (P.). Lehr- und Uebungsbuch der Arithmetik und Algebra für höhere Mädchenschulen. Leipzig, Hirt, 1907. 8vo. 196 pp. M. 2.00

ROBBINS (E. R.). Plane and solid geometry. New York, American Book Co., 1907. 8vo. 5 + 412 pp. Cloth. $1.25

——. Solid geometry. New York, American Book Co., 1907. 8vo. Pp. 15 + 251 — 409. Cloth. $0.75

ROMANO (F.). Un teorema sulla teoria delle progressioni aritmetiche di ordine *k*. Avola, Piazza, 1907. 16mo. 8 pp.

RUSSELL (J. W.). A sequel to elementary geometry, with numerous examples. London, Frowde, 1907. 8vo. 212 pp. 6s.

SALOMON (Mme A.). Leçons d'algèbre, à l'usage de l'enseignement secondaire des jeunes filles (Classes de quatrième et cinquième années) et des aspirantes au brevet supérieur. 4e édition. Paris, Vuibert, 1908. 16mo. 190 pp. Fr. 2.00

SANFELICI (G.). Calcolo tacheometrico semplificato. Tavole a graduazione centesimale e sessuagenerie, arricchite dei logaritmi dei numeri e delle funzioni trigonometriche. 2a edizione in Italiano e Francese. Milano, 1907. 4to. 270 pp. M. 10.50

SCHWERING (K.). Trigonometrie für höhere Lehranstalten. Nach den amtlichen Lehrvorschriften bearbeitet. 3te Auflage. Freiburg, Herder, 1907. 8vo. 7 + 55 pp. M. 1.30

SIEMON (P.). See OTTO (F.).

SLAUGHT (H. E.) and LENNES (N. J.). High school algebra, elementary course. Boston, Allyn and Bacon, 1907. 12mo. 12 + 297 pp. Cloth. $1.00

TORRENTS y MONNER (A.). Pequeñeces matematicas. Barcelona, 1907. 4to. 6 pp. M. 1.00

WEBER (H.) und WELLSTEIN (J.). Encyklopädie der Elementar-Mathematik. Ein Handbuch für Lehrer und Studierende. (In 3 Bänden.) 3ter Band: Angewandte Elementar-Mathematik. Bearbeitet von H. Weber, J. Wellstein und R. H. Weber. Leipzig, Teubner, 1907. 8vo. 13 + 666 pp. Cloth. M. 14.00

WEBER (R. H.). See WEBER (H.).

WELLSTEIN (J.). See WEBER (H.).

WILK (E.) und HAASE (E.). Anweisung zur Geometrie der Mittelschule. 1er Teil. Dresden, Bleyl, 1907. 8vo. 3 + 81 pp. M. 1. 85

III. APPLIED MATHEMATICS.

BACHELET (C.). Alcune considerazioni sulla trattazione della meccanica independentemente dal postulato V di Euclide. I. Torino, 1907. 8vo. 15 pp.

BAKER (W. M.). A key to elementary dynamics. London, Bell, 1907. 8vo. 10s. 6d.

BANDINI (S.). Sui momenti quadratici di un sistema magnetico di punti. Padova, Prosperini, 1907. 8vo. 32 pp.

BAST (O. DE). Eléments du calcul et de la mesure des courants alternatifs. Paris, Béranger, 1907. 8vo. 197 pp.

BAZARD. Mécanique théorique. Cours professé aux écoles nationales d'arts et métiers (4 volumes). Vol. I. Paris, 1906. 8vo. 507 pp. M. 8.50

BORN (M.). Untersuchungen über die Stabilität der elastischen Linie in Ebene und Raum, unter verschiedenen Grenzbedingungen. (Diss). Göttingen, 1906. 8vo. 101 pp.

CAPPA (S.). Corso di idraulica pratica. Torino, Pasta, 1907. 8vo. 124 + 1055 pp. L. 32.00

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THE SEPTEMBER MEETING OF THE SAN FRANCISCO SECTION.

THE twelfth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of California on Saturday, September 28, 1907. The following members of the Society were present:

Professor R. E. Allardice, Professor H. F. Blichfeldt, Mr. A. J. Champreux, Professor R. L. Green, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Mr. Joseph Lipke, Dr. J. H. McDonald, Professor W. A. Manning, Dr. T. M. Putnam, Professor Irving Stringham, Mr. J. D. Suter.

The following officers were elected for the ensuing year: Professor L. M. Hoskins, chairman; Professor W. A. Manning, secretary; Professor D. N. Lehmer, Dr. J. H. McDonald, Professor W. A. Manning, program committee. The next two meetings of the Section are to be held at Stanford University, February 29, 1908, and at the University of California, September 26, 1908.

The following papers were read at this meeting:

(1) Dr. J. H. MCDONALD: "On Minkowski's diagonal continued fraction."

(2) Professor H. F. BLICHFELDT: "Concerning the geodesics connecting five points."

(3) Professor D. N. LEHMER: "A report on Kulik's manuscript tables of factors."

(4) Professor D. N. LEHMER: "Extension of certain theorems in the theory of quadratic residues."

(5) Professor C. A. NOBLE: "Singular points of a simple kind of a differential equation of the second order."

(6) Professor G. A. MILLER: "Groups in which the subgroup which involves all the substitutions omitting a given letter is regular."

(7) Professor R. E. ALLARDICE: "Note on the cyclide of Dupin."

(8) Dr. J. H. MCDONALD: "On the condition that two circles may have a simultaneously in- and circumscribed quadrilateral."

(9) Professor L. E. DICKSON: "On quadratic forms in a general field."

The papers of Professors Noble, Miller and Dickson were read by title. Abstracts of the papers are given below in order as numbered in the list above.

1. Minkowski has drawn attention to the properties of a type of continued fraction development of a quantity x which is such that if p/q is a convergent, $|p-xq| < 1/2q$, and conversely if $|p-xq| < 1/2q$, p/q is a convergent. Minkowski's method is the use of a geometric representation. It is the object of Dr. McDonald's communication to show how to convert the usual development for x by a simple transformation into this diagonal continued fraction.

2. Professor Blichfeldt proved that a surface of revolution is the only surface possessing the following property: Five arbitrary points on the surface being selected, the ten geodesic distances connecting these points satisfy a relation independent of the coordinates of the five points.

3. Assisted by the Carnegie Institution of Washington, Professor Lehmer has made a careful comparison of his own tables with those previously existing. The first, second and third millions were compared, entry for entry, with Burckhardt's, the fourth, fifth and sixth with Glaisher's, the seventh, eighth and ninth with Dase's. The comparison will be made a second time and a list of errors published.

For the tenth million, Professor Lehmer hoped to avail himself of the manuscript deposited by Rosenberg's widow with the Berlin Academy. Unfortunately, these manuscripts seem to have disappeared. The manuscript tables of Professor Kulik, left in charge of the Vienna Academy, were however available, and with the generous permission of the heirs of Professor Kulik were obtained for examination and comparison.

The Kulik tables are remarkable for their extraordinary extent—to 100,330,201, according to the title page of the first of the six volumes (only one of the volumes was obtained for examination), and for the ingenious table of abbreviations. Primes up to 163 are represented by a single character: $a = 11$, $b = 13$, $c = 17$, etc., to $z = 109$. Then the digits are used (except 7, which was printed into the table as usual): $1 = 113$, $2 = 127$, $\cdots$, $9 = 157$, $0 = 163$. Zero and the letter o are distinguished in the manuscript by a stroke through the former. From 167 on, the primes are given by double symbols: $aa = 167$, $ba = 173$, etc. By using for the second letter all the letters of the Roman alphabet (j excepted) and all the letters of the German alphabet up to o, the primes are covered as far as 8057.

The arrangement of the table is the same as Burckhardt's. Each page of the manuscript contains 77 columns. The "sieve method" of making the entries was used for primes as high as 1009, after that the "multiple method" was used.

Something like two hundred errors were found in the manuscript for the tenth million. The exact number will be given after the second comparison.

4. Defining a quadratic residue as any number that is congruent to a square, modulo m, and leaving out the usual restriction that the number shall be prime to the modulus, Professor Lehmer obtains in his second paper formulas for the number of distinct residues of this sort.

It is a known theorem that for a prime of the form $4n \pm 1$ there will be found exactly n pairs of quadratic residues that differ by unity. This theorem was also extended to residues as defined above.

5. Professor Noble's paper will appear in full in the BULLETIN.

6. As the groups in which the regular subgroup G_1 is of degree $n - 1$, n being the degree of the group G, have received considerable attention, Professor Miller considers mainly those groups in which the degree of G_1 is $n - \alpha$ $(\alpha > 1)$. Some of the main results may be stated as follows: G contains n/α systems of imprimitivity which it permutes either according to the group of order 2 or according to a multiply transitive group. When G_1 is also abelian, G contains exactly α^2 substitutions which transform each of these systems into itself, and it contains an additional invariant subgroup of index $n/\alpha - 1$. The latter of these two invariant subgroups includes the former whenever $n/\alpha > 2$. In the special case when $n/\alpha = 2$ this subgroup reduces to identity. To arrive at these results frequent use was made of the following theorems: If a regular group H of degree n is transformed into itself by a substitution s of degree $n - \alpha$ in the same letters, then s is commutative with exactly α of the substitutions of H whenever $\alpha > 0$. When $\alpha = 0$,

s must be commutative with at least one of the substitutions of H besides identity. If a transitive group of degree n is transformed into itself by any substitution in the same letters and if the degree of this substitution is not $n - 1$, then it must be commutative with at least one of the substitutions of the transitive group besides identity.

7. The object of Professor Allardice's note on the cyclide of Dupin was to show that, by means of a transformation originally due to Laguerre (see Darboux, Théorie des surfaces, volume 1, page 253), a circle may be transformed into this cyclide; and that the principal properties of the surface may be obtained geometrically by means of the transformation.

8. The relation between the radii and distance between centers giving the condition that two circles may have a simultaneously in- and circumscribed quadrilateral was obtained by various mathematicians (Fuss, Steiner, Jacobi, Cayley) in a form limited to a special case. The complete formulas are found by Dr. McDonald, incidentally giving the interpretation of certain results of the theory of elliptic functions.

9. Professor Dickson's paper appears in full in the present number of the BULLETIN.

W. A. MANNING,
Secretary of the Section.

ON QUADRATIC FORMS IN A GENERAL FIELD.

BY PROFESSOR L. E. DICKSON.

(Read before the San Francisco Section of the American Mathematical Society, September 28, 1907.)

1. WE investigate the equivalence, under linear transformation in a general field F, of two quadratic forms*

$$q \equiv \sum_{i=1}^{n} a_i x_i^2, \quad Q \equiv \sum_{i=1}^{n} \alpha_i X_i^2 \quad (a_i \neq 0, \alpha_i \neq 0).$$

An obvious necessary condition is that α_1 shall be representable by q, viz., that there shall exist elements b_i in F such that

$$\alpha_1 = \sum_{i=1}^{n} a_i b_i^2.$$

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* Within any field F, not having modulus 2, any quadratic form of non-vanishing determinant is equivalent to one of type q.

We assume that this condition is satisfied. Applying a suitable permutation of the x_i, we may set (§ 2)

$$W_k \equiv \sum_{i=1}^{k} a_i b_i^2 \neq 0 \qquad (k = 1, \cdots, n).$$

In view of § 3, the transformation

$$x_i = b_i y_1 + W_{i-1} y_i - b_i \sum_{j=i+1}^{n} a_j b_j y_j \quad (i = 1, \cdots, n)$$

is of non-vanishing determinant and replaces q by

$$q' \equiv \alpha_1 y_1^2 + \sum_{j=2}^{n} a_j W_j W_{j-1} y_j^2.$$

By the theorem proved in § 4 by a consideration of the automorphs of q', the forms Q and q' (with like first coefficients) are equivalent in F if, and only if,

$$(1) \qquad \sum_{i=2}^{n} \alpha_i X_i^2 = \sum_{j=2}^{n} a_j W_j W_{j-1} y_j^2$$

under a transformation in F on $n - 1$ variables. Hence *the necessary and sufficient conditions for the equivalence of the two given n-ary quadratic forms q and Q are that α_1 be representable by q and that the $(n-1)$-ary forms* (1) *be equivalent in F.* The ultimate criteria are that $\alpha_1, \alpha_2, \cdots, \alpha_n$ be representable by forms in $n, n-1, \cdots, 1$ variables, respectively, whose coefficients are given functions of the a_i. For example, if $n = 2$, the conditions are that α_1 be representable by q and that $a_1 a_2 \alpha_1 \alpha_2$ be a square in F. If $n = 3$, the conditions are that α_1 be representable by q, α_2 by $a_1 a_2 W_2 \xi^2 + a_3 \alpha_1 W_2 \eta^2$, and that $a_1 a_2 a_3 \alpha_1 \alpha_2 \alpha_3$ be a square in F.

2. THEOREM.* *In a non-modular*† *field F there exists a form*

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[*] *Transactions*, vol. 7 (1906), pp. 276–8. The present proof is decidedly simpler and leads to the explicit expressions (17) for the A_j, as required for the present applications.

[†] The proof applies to fields having a modulus p, where $p \neq 2$ and $p \geqq n$. It may be extended to apply to any finite field. To q we apply the transformation

$$x_1 = r y_1 + s y_2, \quad x_2 = t y_1 - t^{-1} a_2^{-1} r s a_1 y_2 \qquad (t \neq 0, s \neq 0),$$

and obtain $m y_1^2 + m a_1 s^2 a_2^{-1} t^{-2} y_2^2 + a_3 x_3^2 + \cdots$, where $m = a_1 r^2 + a_2 t^2 \neq 0$. By choice of r and t, we may give m any assigned value in a finite field.

$\sum A_i y_i^2$ equivalent to a given form $q = \sum a_i x_i^2$ and having as its first coefficient A_1 any preassigned mark $\neq 0$ which is representable by q.

By hypothesis, there exist marks b_{i1} of F such that

$$(2) \qquad \sum_{i=1}^{n} a_i b_{i1}^2 = A_1 \neq 0.$$

Not every sum of $n-1$ of the terms t_i of (2) vanishes. For if so, we consider the sum lacking t_j and the sum lacking t_k and conclude that $t_j = t_k$ and hence that the n terms are all equal and that each is not zero; but this requires that F shall have a modulus dividing $n-1$. By applying a permutation on the x's in q, we may set $\sum_{i=1}^{n-1} a_i b_{i1}^2 \neq 0$. Not every sum of $n-2$ terms of the latter vanishes, since F does not have a modulus dividing $n-2$, etc. We may therefore set

$$(3) \qquad W_k \equiv \sum_{i=1}^{k} a_i b_{i1}^2 \neq 0 \qquad (k = 1, \cdots, n).$$

Under the transformation

$$(4) \qquad x_i = \sum_{j=1}^{n} b_{ij} y_j \qquad (i = 1, \cdots, n),$$

q becomes $\sum (A_j y_j^2 + 2B_{jk} y_j y_k)$, $(j, k = 1, \cdots, n; k > j)$, where

$$(5) \qquad A_j = \sum_{i=1}^{n} a_i b_{ij}^2, \quad B_{jk} = \sum_{i=1}^{n} a_i b_{ij} b_{ik}.$$

To make $B_{1k} = 0$, we take

$$(6) \qquad b_{1k} = -a_1^{-1} b_{11}^{-1} \sum_{i=2}^{n} a_i b_{i1} b_{ik} \qquad (k = 2, \cdots, n).$$

We insert these values in $\Delta = |b_{ij}|$, remove the factor $a_1^{-1} b_{11}^{-1}$ from the first row, then multiply the ith row by $a_i b_{i1}$ and add to the first row, for $i = 2, \cdots, n$. We get

$$(7) \qquad \Delta = a_1^{-1} b_{11}^{-1} A_1 \Delta_{11}, \quad \Delta_{11} \equiv |b_{is}| \quad (i, s = 2, \cdots, n).$$

We introduce the abbreviations, in addition to (3),

$$(8) \qquad P_{it} = a_i a_t b_{i1} b_{t1}, \quad R_{si} = a_i (a_i b_{i1}^2 + W_{s-1}),$$

so that, in particular, $R_{ss} = a_s W_s$. We find that

$$(9) \quad R_{ss} R_{si} - P_{is}^2 = a_s W_{s-1} R_{s+1 i}, \quad R_{ss} P_{it} - P_{is} P_{st} = a_s W_{s-1} P_{it}.$$

Eliminating the b_{1k} from $a_1 b_{11}^2 B_{jk} (2 \leqq j < k)$ by (6), we get

$$(10) \qquad B_{jk}^{(2)} \equiv \sum_{i=2}^{n} b_{ij} \left(R_{2i} b_{ik} + \sum_{t \neq i}^{t=2,\ldots,n} P_{it} b_{tk} \right).$$

Now $R_{22} \equiv a_2 W_2 \neq 0$. To make the coefficient of b_{2j} zero, we take

$$(11) \qquad b_{2k} = - R_{22}^{-1} \sum_{t=3}^{n} P_{2t} b_{tk} \qquad (k = 3, \cdots, n).$$

Set $b_{i2} = 0 \; (i > 2)$. Then $B_{2k}^{(2)}$ evidently vanishes, while

$$(12) \qquad \Delta_{11} = b_{22} \Delta_{22}, \quad \Delta_{22} \equiv | b_{is} | \qquad (i, s = 3, \cdots, n).$$

Generalizing (6) and (11), we shall take

$$(13) \qquad b_{sk} = - R_{ss}^{-1} \sum_{t=s+1}^{n} P_{st} b_{tk} \quad (k = s + 1, \cdots, n),$$

$$(14) \qquad b_{il} = 0 \qquad (i > l > 1),$$

and then prove by induction from s to $s + 1$ that, for

$$1 < s \leqq j < k \leqq n,$$

the product of B_{jk} by a non-vanishing factor equals

$$(15) \qquad B_{jk}^{(s)} \equiv \sum_{i=s}^{n} b_{ij} \left(R_{si} b_{ik} + \sum_{t \neq i}^{t=s,\ldots,n} P_{it} b_{tk} \right).$$

By (10) this statement is true for $s = 2$. The coefficient of b_{sj} in (15) is zero by (13), so that we may set $i > s$. From the term given by $t = s$, we eliminate b_{sk} by means of (13). Hence

$$R_{ss} B_{jk}^{(s)} = \sum_{i=s+1}^{n} b_{ij} \left\{ (R_{ss} R_{si} - P_{is}^2) b_{ik} + \sum_{t \neq i}^{t=s+1,\ldots,n} (R_{ss} P_{it} - P_{is} P_{st}) b_{tk} \right\}.$$

Applying (9), we get

$$R_{ss} B_{jk}^{(s)} = a_s W_{s-1} B_{jk}^{(s+1)},$$

so that the induction is complete. Thus for given values of j and k, $1 < j < k$, we may increase s and make $s > j$; then (15) vanishes by (14). Hence every $B_{jk} = 0 \; (k > j)$. Finally, if we take each $b_{ii} \neq 0$, the determinant of (4) is not zero, in view of (14) and (7) or (12).

3. From (13), in combination with (14) and (9_2), we get

$$b_{sk} = -a_s^{-1} W_{k-1}^{-1} P_{sk} b_{kk} \qquad (s < k). \tag{16}$$

Starting with the simple formulas (14) and (16), we may readily verify that, in (5), each $B_{jk} = 0$, and

$$A_j = a_j W_j W_{j-1}^{-1} b_{jj}^2 \qquad (j = 2, \cdots, n). \tag{17}$$

THEOREM. *For given elements b_{i1} satisfying* (2) *and* (3), *and any elements $b_{jj} \neq 0$ $(j > 1)$, the transformation, of non-vanishing determinant,*

$$x_i = b_{i1} y_1 + b_{ii} y_i - b_{i1} \sum_{j=i+1}^{n} a_j b_{j1} b_{jj} W_{j-1}^{-1} y_j \qquad (i = 1, \cdots, n) \tag{18}$$

replaces $\sum a_i x_i^2$ by $\sum A_j y_j^2$, the A_j being given by (2), (17).

If we employ the special values $b_{jj} = W_{j-1}$ and set $b_{i1} = b_i$, we obtain the simpler results given in § 1.

4. Within the field F, let the forms

$$Q = \sum_{i=1}^{n} a_i X_i^2, \quad E = \sum_{i=1}^{n} e_i y_i^2 \qquad (a_i \neq 0,\ e_i \neq 0) \tag{19}$$

be equivalent under the transformation

$$S\colon \quad X_i = \sum_{j=1}^{n} \sigma_{ij} y_j \qquad (i = 1, \cdots, n). \tag{20}$$

In view of the formulas

$$2e_i y_i = \frac{\partial E}{\partial y_i} = \frac{\partial Q}{\partial y_i} = \sum_{j=1}^{n} 2a_j X_j \frac{\partial X_j}{\partial y_i} = 2\sum_{j=1}^{n} a_j \sigma_{ji} X_j,$$

the inverse of S is

$$S^{-1}\colon \quad y_i = e_i^{-1} \sum_{j=1}^{n} a_j \sigma_{ji} X_j \qquad (i = 1, \cdots, n). \tag{21}$$

Eliminating the y_i from $Q = E$, we get

$$\sum_{i=1}^{n} e_i^{-1} \sigma_{ji}^2 = a_j^{-1}, \quad \sum_{i=1}^{n} e_i^{-1} \sigma_{ji} \sigma_{ki} = 0 \qquad (j \neq k). \tag{22}$$

Next, let $E(y) = mE(\xi)$ under the transformation

$$(23) \qquad C: \quad \xi_i = \sum_{j=1}^{n} \gamma_{ij} y_j \qquad (i = 1, \cdots, n).$$

Replacing α_j by me_j and σ_{ji} by γ_{ji} in (21) and (22), we get

$$(24) \qquad C^{-1}: \quad y_i = me_i^{-1} \sum_{j=1}^{n} e_j \gamma_{ji} \xi_j \qquad (i = 1, \cdots, n),$$

$$(25) \qquad \sum_{i=1}^{n} e_i^{-1} \gamma_{ji}^2 = m^{-1} e_j^{-1}, \quad \sum_{i=1}^{n} e_i^{-1} \gamma_{ji} \gamma_{ki} = 0 \qquad (j \neq k).$$

Eliminating the y_i between (20) and (24), we obtain relations of the form

$$(26) \qquad X_i = \sum_{j=1}^{n} \beta_{ij} \xi_j \qquad (i = 1, \cdots, n),$$

which imply $Q = mE(\xi)$. We desire that the first of these relations shall reduce to $X_1 = \xi_1$. By (20), (23), the conditions are

$$(27) \qquad \gamma_{1j} = \sigma_{1j} \qquad (j = 1, \cdots, n).$$

In view of (22_1) and (25_1), for $j = 1$, these require that $me_1 = \alpha_1$. Conversely, if $Q = E$ under S, and if there exists in F a matrix (γ_{ij}) with the first row identical with that of S and such that $E(y) = e_1^{-1}\alpha_1 E(\xi)$ under C, then will $Q = e_1^{-1}\alpha_1 E(\xi)$ under a transformation (26) with $X_1 = \xi_1$, and therefore

$$(28) \qquad \sum_{i=2}^{n} \alpha_i X_i^2 = e_1^{-1} \alpha_1 \sum_{i=2}^{n} e_i \xi_i^2,$$

under a transformation in F on $n - 1$ variables.*

As noted in § 1, it suffices to treat the case † $e_1 = \alpha_1$. We have therefore to deal with automorphs of $E(y)$. In view of

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* For $\sigma_{1i} = 0$ $(i > 1)$, (22_2) with $j = 1$ gives $\sigma_{k1} = 0$ $(k > 1)$.

† The general case presents an essential difficulty. By (24), $|C^{-1}| = m^n |C|$, so that $m^n = \Delta^{-2}$, where $\Delta = |\gamma_{ij}|$. For n odd, m must therefore be a square in F, and hence cannot, in general, be made equal to $e_1^{-1}\alpha_1$, where α_1^{-1} is an arbitrary element of the form $\Sigma e_i^{-1}\sigma_{1i}^2$; see (22_1) for $j = 1$. The case $n = 2$ is quite simple; we may take as (γ_{ij}) the matrix

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ -e_2^{-1}e_1\sigma_{12} & \sigma_{11} \end{pmatrix}, \quad e_1^{-1}\sigma_{11}^2 + e_2^{-1}\sigma_{12}^2 = \alpha_1^{-1}.$$

§ 5, we can determine a matrix (γ_{ij}) satisfying the above conditions. We therefore have the

THEOREM. *If Q and E are equivalent in F under S and if $e_1 = \alpha_1$, then Q is equivalent to $E(\xi)$ under a transformation in F with $X_1 = \xi_1$, so that $\sum_{i=2}^{n} \alpha_i X_i^2$ and $\sum_{i=2}^{n} e_i \xi_i^2$ are equivalent in F under a transformation on $n - 1$ variables.*

Conversely, if the latter forms are equivalent in F, then evidently Q and E, with $e_1 = \alpha_1$, are equivalent in F.

5. THEOREM. *For any set of solutions σ_i in F of*

$$\sum_{i=1}^{n} e_i^{-1}\sigma_i^2 = e_1^{-1}, \tag{29}$$

the quadratic form $E = \sum e_i y_i^2$ has an automorph in F which replaces y_1 by $\sum \sigma_j y_j$.

Let Y be any skew symmetric matrix, I the identity (unit) matrix, E^{-1} the inverse of the matrix of E. Let $Z = E^{-1}Y$. Then, by Cayley's theorem, E has the automorph

$$P \equiv (I + Z)^{-1}(I - Z) \quad (\Delta \equiv |I + Z| \neq 0).$$

It will suffice to take as Y the skew matrix in which the elements outside of the first row and column are all zero, while the first row is

$$0,\ c_2,\ \cdots,\ c_n.$$

Hence

$$I \pm Z = \begin{pmatrix} 1 & \pm c_2 e_1^{-1} & \pm c_3 e_1^{-1} & \cdots & \pm c_n e_1^{-1} \\ \mp c_2 e_2^{-1} & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mp c_n e_n^{-1} & 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$\Delta = 1 + \kappa, \quad \kappa \equiv e_1^{-1} \sum_{i=2}^{n} e_i^{-1} c_i^2. \tag{30}$$

The first row of $(I + Z)^{-1}$ is

$$\frac{1}{\Delta},\ \frac{-c_2}{e_1\Delta},\ \frac{-c_3}{e_1\Delta},\ \cdots,\ \frac{-c_n}{e_1\Delta}.$$

Hence the first row of the product P is

$$\frac{1-\kappa}{\Delta},\ \frac{-2c_2}{e_1\Delta},\ \frac{-2c_3}{e_1\Delta},\ \cdots,\ \frac{-2c_n}{e_1\Delta}.$$

These elements are to be made equal to

$$\sigma_1,\ \sigma_2,\ \sigma_3,\ \cdots,\ \sigma_n,$$

respectively. Hence, by (30_1), we take

$$2=\Delta(1+\sigma_1),\quad \kappa=\tfrac{1}{2}\Delta(1-\sigma_1),\quad c_i=-\tfrac{1}{2}\Delta e_1\sigma_i\quad (i=2,\cdots,n).$$

Eliminating the c_i from (30_2), and applying (29), we get

$$\kappa = \tfrac{1}{4}\Delta^2 e_1 \sum_{i=2}^{n} e_i^{-1}\sigma_i^2 = \tfrac{1}{4}\Delta^2(1-\sigma_1^2).$$

Hence the conditions may be satisfied if $1+\sigma_1 \neq 0$. But for $\sigma_1 = -1$, we may first apply the automorph $y_1' = -y_1$.

6. Although not employed in the present paper, the following generalization of the preceding theorem may be noted:

THEOREM. *Let r be any positive integer* $\leqq n$. *If the* σ_{ji} *are any solutions in F of relations* (22), *for* $j, k = 1, \cdots, r$, *and* $a_j = e_j$, *the form E has an automorph in F which replaces* y_i *by* $\sum_{j=1}^{n} \sigma_{ij}y_j$, *for* $i = 1, \cdots, r$.

The proof by induction, based on the result of § 5, is similar to that in the *American Journal*, volume 23 (1901), page 344, for the special case of a finite field with special values of the e_i.

7. Let F be field R of all rational numbers. There exist[*] rational values of $b_1, \cdots, b_4$ such that $\sum_{i=1}^{4} a_i b_i^2$ equals $+1$ or -1, according as $a_1, \cdots, a_4$ are not all negative, or all negative. Hence, by § 1, any n-ary rational quadratic form of non-vanishing determinant is reducible by a linear transformation with rational coefficients to one of the forms

$$f_{p,a,b,c} \equiv \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{n-3} x_i^2 + ax_{n-2}^2 + bx_{n-1}^2 + cx_n^2,$$

in which a, b, c are all negative if $p < n-3$, while $f_{p,a,b,c}$ is reducible to $f_{p,\alpha,\beta,\gamma}$ if, and only if,[†] the ternary form

$$t_{a,b,c} \equiv ax^2 + by^2 + cz^2$$

is reducible in R to $t_{\alpha,\beta,\gamma}$ (see end of § 1).

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[*] A simple consequence (*Transactions*, l. c., p. 279) of a theorem due to A. Meyer; cf., Bachmann, Zahlentheorie, IV_1, p. 266.

[†] This part of the result was not given in my former paper.

ON THE CANONICAL SUBSTITUTION IN THE HAMILTON-JACOBI CANONICAL SYSTEM OF DIFFERENTIAL EQUATIONS.

BY DR. D. C. GILLESPIE.

(Read before the American Mathematical Society, September 5, 1907.)

LIE established a one-to-one correspondence between the integrals of the canonical system of differential equations and the one-parameter continuous groups of contact transformations of which the system admits, *i. e.*, making use of an integral of the system one can construct the infinitesimal transformation of a group of which the system admits, or, on the other hand by making use of the infinitesimal transformation of a group of which the system admits one can construct an integral of the system.* This theorem is the foundation of the modern transformation theory of dynamical systems.† The single canonical substitution (introduced by Jacobi) is of importance in the transformation and simplification of the dynamical equations.‡

The purpose of this paper is to define a type of the single canonical substitution which leads to an integral of the equations, *i. e.*, if one knows a member of the defined type of canonical substitutions one can construct an integral of the system using only algebraic operations.

A system of differential equations which has the form

$$(1)\quad \frac{d\pi}{dx}=\frac{\partial H}{\partial y},\quad \frac{dy}{dx}=-\frac{\partial H}{\partial \pi},\quad \frac{d\kappa}{dx}=\frac{\partial H}{\partial z},\quad \frac{dz}{dx}=-\frac{\partial H}{\partial \kappa}$$

is called a canonical system. H is a known function of y, π, z, κ and x; y, π, z and κ are the unknown functions; thus the solution of the system of four equations (1) consists in determining the four unknowns y, π, z and κ as such functions of x that the equations become identities in x. The functions y and π, as also z and κ, are called conjugate.

A substitution which leaves the form of the system (1) unchanged, though the function H may or may not be changed, is a canonical substitution.

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* Whittaker, Analytical Dynamics, page 308.

† Ibid., page 292.

‡ Poincaré, Mécanique céleste ; Jacobi, Vorlesungen über Dynamik.

In order to avoid complicated formulas I take a system of four equations. The generalization of the theory from four to $2n$ equations is evident.

The system of equations (1) expresses the necessary condition that the integral

$$\int_1^2 \{\pi y' + \kappa z' + H(y, \pi, z, \kappa, x)\} dx \tag{2}$$

(where $dy/dx \equiv y'$, $dz/dx \equiv z'$, and 1 and 2 stand for any limits of integration) take a maximum or a minimum value; *i. e.*, if the question be, what functions of x must y, π, z and κ be in order that the integral (2) shall take an extreme value, the answer is, only such functions as satisfy the equations (1).

In order that y and z be such functions of x that the integral

$$\int_1^2 F(y', y, z', z, x) dx \tag{3}$$

take an extreme value, it is necessary that y and z satisfy the system of equations

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0, \quad \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) - \frac{\partial F}{\partial z} = 0. \tag{4}$$

If now

$$F(y', y, z', z, x) \equiv H(y, \pi, z, \kappa, x) + y'\pi + z'\kappa, \tag{5}$$

whereby is understood that instead of π and κ are substituted their values in terms of y', y, z', and z, determined from the equations

$$y' = -\frac{\partial H(y, \pi, z, \kappa, x)}{\partial \pi}, \quad z' = -\frac{\partial H(y \cdot \pi, z, \kappa, x)}{\partial \kappa}, \tag{6}$$

then the two systems (4) and (1) are equivalent. In fact let

$$\begin{aligned} y &= y(x, c_1, c_2, c_3, c_4), & z &= z(x, c_1, c_2, c_3, c_4), \\ \pi &= \pi(x, c_1, c_2, c_3, c_4), & \kappa &= \kappa(x, c_1, c_2, c_3, c_4) \end{aligned} \tag{7}$$

(c_1, c_2, c_3, c_4 are arbitrary constants) be the complete solution of (1), then

$$y = y(x, c_1, c_2, c_3, c_4), \quad z = z(x, c_1, c_2, c_3, c_4) \tag{8}$$

is the complete solution of (4). Both systems of equations are satisfied by the same values of y and z. Conversely now let

$$y = y(x, c_1, c_2, c_3, c_4), \quad z = z(x, c_1, c_2, c_3, c_4)$$

be the complete solution of (1), then

$$y = y(x, c_1, c_2, c_3, c_4), \qquad z = z(x, c_1, c_2, c_3, c_4),$$

$$(9) \quad \pi = \frac{\partial}{\partial y'} F(y', y, z', z, x), \quad \kappa = \frac{\partial}{\partial z'} F(y', y, z', z, x)$$

is the complete solution of (4). Thus the problems of integrating these two systems of equations are equivalent.

We will now suppose a canonical substitution known for the system of differential equations (4),

$$(10) \quad \pi_1 = f_1(y, \pi, z, \kappa, x), \ \kappa_1 = f_2(y, \pi, z, \kappa, x), \ z_1 = f_3(y, \pi, z, \kappa, x),$$
$$y_1 = f_4(y, \pi, z, \kappa, x), \quad x_1 = f_5(y, \pi, z, \kappa, x).$$

Under this substitution the $H(y, \pi, z, \kappa, x)$ goes over into another function $H^*(y_1, \pi_1, z_1, \kappa_1, x_1)$, but the form of the equations remains canonical. We wish to impose a condition that will fix the substitution (10) as a member of the type of canonical substitutions which leads to an integral of (4). The condition is that the y and z functions of the independent variable x remain unchanged. This means that if

$$y = \phi_1(x, c_1, c_2, c_3, c_4), \quad z = \phi_2(x, c_1, c_2, c_3, c_4),$$
$$\pi = \phi_3(x, c_1, c_2, c_3, c_4), \quad \kappa = \phi_4(x, c_1, c_2, c_3, c_4)$$

is the complete solution of (4), then

$$y_1 = \phi_1(x_1, c_1, c_2, c_3, c_4), \quad z_1 = \phi_2(x_1, c_1, c_2, c_3, c_4),$$
$$\pi_1 = \omega_1(x_1, c_1, c_2, c_3, c_4), \quad \kappa_1 = \omega_2(x_1, c_1, c_2, c_3, c_4)$$

(where ω_1 and ω_2 mean simply functions different from ϕ_3 and ϕ_4) is the complete solution of the system of equations obtained by subjecting the system (1) to the substitution (10).

In order to determine whether a given canonical substitution belongs to the defined type, it would seem necessary to know the complete solutions both of the system of equations (1) and the system obtained by subjecting (1) to the given substitution. This however is not the case. For suppose again that (10) is any given canonical substitution of which we wish to determine whether it belongs to the defined type. Transforming the system of equations (1) by (10), we obtain a new function

$$H^*(y_1, \pi_1, z_1, \kappa_1, x)$$

and hence a new system of differential equations

$$(11)\qquad \begin{aligned} \frac{d\pi_1}{dx_1} &= \frac{\partial H^*}{\partial y_1}, & \frac{d\kappa_1}{dx_1} &= \frac{\partial H^*}{\partial z_1}, \\ \frac{dy_1}{dx_1} &= -\frac{\partial H^*}{\partial \pi_1}, & \frac{dz_1}{dx_1} &= -\frac{\partial H^*}{\partial \kappa_1}. \end{aligned}$$

Making use of this new function $H^*(y_1, \pi_1, z_1, \kappa_1, x_1)$, we form a new function $F^*(y_1', y_1, z_1', z_1, x_1)$, just as in (5),

$$F^*(y_1', y_1, z_1', z_1, x) \equiv H^*(y_1, \pi_1, z_1, \kappa_1, x_1) + y_1'\pi_1 + z_1'\kappa,$$

where instead of π_1 and κ_1 are substituted their values in terms of y_1', y_1, z_1' and z_1 determined from the equations

$$y_1' = -\frac{\partial H^*(y_1, \pi_1, z_1, \kappa_1, x_1)}{\partial \pi_1}, \quad z_1' = -\frac{\partial H^*(y_1, \pi_1, z_1, \kappa_1, x_1)}{\partial \kappa_1}.$$

We express now the necessary condition that the integral

$$(12)\qquad \int_1^2 F^*(y_1', y_1, z_1', z_1, x_1)dx_1$$

take an extreme value and obtain the equations

$$(13)\quad \frac{d}{dx_1}\left(\frac{\partial F^*}{\partial y_1'}\right) - \frac{\partial F^*}{\partial y_1} = 0, \quad \frac{d}{dx_1}\left(\frac{\partial F^*}{\partial z_1'}\right) - \frac{\partial F^*}{\partial z_1} = 0.$$

The systems of equations (1) and (4) are satisfied by the same functional values of y and z, $y = f_1(x, c_1, c_2, c_3, c_4)$, $z = f_2(x, c_1, c_2, c_3, c_4)$; so also are the systems (11) and (13), $y_1 = f_1^*(x_1, c_1, c_2, c_3, c_4)$, $z_1 = f_2^*(x_1, c_1, c_2, c_3, c_4)$. If now the transformation (10) belongs to the defined type, then $f_1(x, c_1, c_2, c_3, c_4) \equiv f_1^*(x, c_1, c_2, c_3, c_4)$ and

$f_2(x, c_1, c_2, c_3, c_4) \equiv f_2^*(x, c_1, c_2, c_3, c_4)$. This means of course that if we drop the subscripts from x_1, y_1, z_1 in the system (13), it becomes identical with (4). This then is the condition that a given transformation be of the defined type.

Since the system of equations (4) expresses the necessary condition that both the integrals

$$\int_1^2 F(y', y, z', z, x)dx, \qquad \int_1^2 F^*(y', y, z', z, x)dx$$

take an extreme value,*

$$\begin{vmatrix} F_{y'y'} & F_{y'z'} \\ F_{z'y'} & F_{z'z'} \end{vmatrix} \div \begin{vmatrix} F^*_{y'y'} & F^*_{y'z'} \\ F^*_{z'y'} & F^*_{z'z'} \end{vmatrix} = \text{const.} \quad \left(F_{yy'} \equiv \frac{\partial^2 F}{\partial y^2}, \text{ etc.} \right)$$

is an integral of the system (4). This, as we have seen, is equivalent to the construction of an integral of the system (1).

Example. As a simple example where the systems consist of only two equations, we take

$$\text{(1)} \qquad \frac{d\pi}{dx} = 0, \quad \frac{dy}{dx} = -\pi;$$

here $H \equiv \frac{1}{2}\pi^2$, $F \equiv H + y'\pi \equiv \frac{1}{2}\pi^2 + y'\pi$, or since by (1) $y' = -\pi$, $F \equiv -\frac{1}{2}y'^2$. The integral $\int_1^2 -\frac{1}{2}y'dx$ has as necessary condition for an extreme value

$$\text{(2)} \qquad y'' = 0.$$

We subject (1) to the substitution

$$\text{(3)} \qquad x = y_1, \quad y = -x_1, \quad \pi = \pi_1,$$

and obtain

$$\text{(4)} \qquad \frac{d\pi_1}{dx_1} = 0, \quad \frac{dy_1}{dx_1} = -\frac{1}{\pi_1},$$

$H^* = \log \pi_1$, $F^* = \log \pi_1 + y'\pi_1 = \log(-1/y_1') - 1$, the necessary condition that the integral $\int_1^2 (\log(-1/y_1') - 1)\,dx$ take an extreme value is

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*D. C. Gillespie, "On the construction of an integral of Lagrange's equations in the calculus of variations," BULLETIN, 2d series, vol. 13, no. 7, pp. 345–348.

$$(5) \qquad y_1'' = 0,$$

the substitution (3) is a member of the defined type and $F_{y'y'} \div F_{yy'} \equiv y' =$ const. is an integral of (2). $y' = c$, $\pi = \partial F/\partial y' = -y'$ gives $\pi = -c$ for an integral of (1).

Cornell University,
August, 1907.

THE MAXIMUM VALUE OF A DETERMINANT.

BY DR. F. R. SHARPE.

Hadamard* has shown that the maximum value of a determinant when the absolute value of each element does not exceed 1 is $n^{\frac{1}{2}n}$. The square of such a maximum determinant is a determinant having all its elements 0 except those of the principal diagonal. If the elements are restricted to real values, they are each ± 1 and are so arranged that when compared row with row there is always an equal number of changes and permanences of sign amongst the corresponding elements. Hence n is necessarily even. If we compare any two rows with a third row, the division of changes and permanences is again even. Hence n must be a multiple of 4. By a rearrangement of signs and order of columns we can always arrange any three rows in the form which for the case of $n = 12$ is

$$\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \end{array}$$

The actual maximum determinant is known for the following cases: (1) n a power of 2, (2) $n = 12$ or 20, (3) when the factors of n are any of the preceding numbers. For example, when n is 8, the determinant is

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* *Bull. des Sciences math.*, 1893.

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{vmatrix} = 4096.$$

The object of this note is to determine the probable maximum value which the determinant actually attains when n is not a multiple of 4, and to prove certain properties of a determinant which does attain the maximum value $n^{\frac{1}{2}n}$ for real values of the elements.

THEOREM I. *The cofactor of any element of a maximum determinant of order* $4p$ *and value* $(4p)^{2p}$ *is* $\pm (4p)^{2p-1}$, *the sign being the same as the sign of that element.*

This follows at once from the form of the square of the determinant and the fact that if we combine the cofactors of one row with the elements of any other row the result is 0.

THEOREM II. *The second minors of a maximum determinant of order* $4p$ *and value* $(4p)^{2p}$ *are* $\pm 2(4p)^{2p-2}$ *or* 0, *according as the complementary minor is* 2 *or* 0.

Arrange the determinant as explained above and consider the minor of the first element of the first row. The cofactors of its first row are clearly 0 for the first $2p - 1$ elements and $2(4p)^{2p-2}$ for the last $2p$ elements, because of the equal distribution of the signs of the elements of the last $2p$ columns.

THEOREM III. *The third minors of a maximum determinant of order* $4p$ *and value* $(4p)^{2p}$ *are* $\pm 4(4p)^{2p-3}$ *or* 0, *according as the complementary minor is* 4 *or* 0.

The determinant being arranged as in the previous theorem, consider the second minor obtained by omitting the first and second rows and the first and $(2p + 1)$th columns. The cofactors of its first row are clearly 0 for the first $p - 1$ elements, $-4(4p)^{2p-3}$ for the next p elements and the last p elements, and 0 for the remaining $p - 1$ elements, because of the equal distribution of signs among the second p and last p columns of the original determinant.

THEOREM IV. *If the sign of any element of a maximum determinant of order* $4p$ *and value* $(4p)^{2p}$ *is changed and its row added to the original determinant together with a suitably chosen column, a determinant of order* $4p+1$ *is formed with a value* $(16p-4)(4p)^{2p-1}$.

Expand the new determinant in terms of the elements of the new column, the signs of which are to be chosen so that the the $4p+1$ determinants obtained are all positive. We obtain the original determinant, the original determinant with a change of sign in one element, and $4p-1$ other determinants with two rows identical except in the sign of one element. Hence the new determinant has the value

$$(4p)^{2p}+(4p)^{2p}-2(4p)^{2p-1}+2(4p-1)(4p)^{2p-1}=(16p-4)(4p)^{2p-1}.$$

Using Theorems I, II and IV, we see that at any rate for $p=1, 2, 3, 4$, or 5 determinants exist

of order $4p-2$, value $2(4p)^{2p-2}$;
of order $4p-1$, value $(4p)^{2p-1}$;
of order $4p$, value $(4p)^{2p}$;
of order $4p+1$, value $(16p-4)(4p)^{2p-1}$.

These give for

$$n=2, 3, 4, 5, 6, 7, 8, \cdots, 16$$

values 2, 4, 16, 48, 128, 512, 4096, $\cdots$, 4 294 967 296.

Professor E. W. Davis's formula* $(n-2)2^{n-1}$ gives for

$$n=3, 4, 5, 6, 7, 8, \cdots, 16$$

values 4, 16, 48, 128, 320, 768, $\cdots$, 458 752.

This formula is clearly too small for $n>6$; in fact, when n is a power of 2 above 5, Hadamard's value is greater than Davis's for the next higher power of 2, and the former's value for $n=24$ is larger than the latter's for $n=2^5$.

CORNELL UNIVERSITY,
October, 1907.

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[*] BULLETIN, October, 1907.

THIRD REPORT ON RECENT PROGRESS IN THE THEORY OF GROUPS OF FINITE ORDER.

BY PROFESSOR G. A. MILLER.

(*Continued from page* 91.)

§ 3. GROUP OF ISOMORPHISMS.

In view of the fundamental importance of the group of isomorphisms of any group it is desirable to have theorems by means of which the group of isomorphisms I of a given substitution group G can be readily determined. If any G of degree n contains n distinct subgroups of degree $n-1$ which are composed of all the substitutions omitting fixed letters, then its substitutions transform these n subgroups just as they transform their own letters and hence G contains no substitution besides identity which is commutative with each one of these n subgroups of degree $n-1$. When G is transitive it will have n such subgroups, provided it has one subgroup of degree $n-1$. Any operator which transforms each of these n subgroups into itself must therefore be commutative with every operator of G. If G does not involve a subgroup of degree n which may correspond to such a subgroup of degree $n-1$ in a holomorphism of G, then its I can be represented as a substitution group of degree n which involves G as an invariant subgroup.

If G is a transitive group and contains a subgroup of degree $n-1$, its I may be represented as a transitive substitution group whose elements are the subgroups which may correspond to one of the largest subgroups of degree $n-1$ in a holomorphism of G. If the degree of this transitive group exceeds n, it must be imprimitive and the m conjugate largest subgroups of degree $n-1$ constitute one system of imprimitivity, while its other systems correspond to subgroups of degree and of index n under G. While these general theorems are frequently directly useful to determine the I of a given G, a number of recent more special theorems find wide application. Among these are the following:

If an abelian group G which involves operators whose orders exceed 2 is extended by means of an operator of order 2 which transforms each operator of G into its inverse, then the I of this extended group is the holomorph of G.* The I of the group obtained by extending the cyclic group of order $2m$ ($m > 2$) by means of an operator of order 4 which transforms each of its operators into its inverse is the holomorph of this cyclic group. The square of a complete group has the double holomorph of this group for its I. The necessary and sufficient condition that a holomorphism corresponds to an invariant operator under I is that the operators which correspond to themselves form an invariant subgroup and that the remaining operators correspond to themselves multiplied by invariant operators.†

If an operator of order 2 in I transforms an operator s of G into s_1s, then it transforms s_1 into its inverse. Hence every operator of order 2 in I transforms some operators of G into their inverses. Moreover, if such an operator transforms every operator of G except identity into a different operator, it must transform every operator of G into its inverse and hence G is an abelian group of odd order. Burnside considered the properties of G when I contains operators of order 3 which transform every operator of G except identity into a different operator. It may be remarked that the review of this note in the *Jahrbuch über die Fortschritte der Mathematik* ‡ is misleading, since it does not state that the operator of order 3 under consideration transforms all of the operators of G except identity into different operators.

By means of the preceding theorems it is easy to find the groups of isomorphisms of substitution groups of low degrees. This has been done for all the groups which can be represented on 7 or a smaller number of letters, as well as for the simple groups whose degrees do not exceed 14. Among the latter the group of order 7920 is especially interesting since it is both complete and simple. It is not difficult to see that the direct product of a complete group which contains only one subgroup of index 2 and the group of order 2 is simply isomorphic with its I. In particular, the direct product of the symmetric group whose degree is not 2 or 6 and the group of order 2 is simply isomorphic with its I.

The groups of isomorphisms of a number of special types of groups of order p^m have recently been determined. Young[*] determined these groups for such groups of order p^m as contain cyclic subgroups of order p^{m-1}. In chapter 2 of Le Vavasseur's work cited above, he considers groups of isomorphisms of the groups of order p^4, and in chapter 3 he considers the groups of cogredient isomorphisms of a large number of the groups of order 2^5. In the current volume of the *Transactions* of this Society, Ranum generalizes the linear congruence groups of Jordan "by using different moduli for the different elements of the matrices, so that each element is a residue of its own particular modulus." He applies these generalized linear congruence groups to the group of isomorphisms of any abelian group and obtains a number of important new theorems. One of these affirms that the necessary and sufficient condition that the group of isomorphisms of an abelian group of order p^m be solvable is that all its invariants are distinct when $p > 3$, and that no three of the invariants are equal to each other when $p = 2$ or 3.[†]

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* *Amer. Jour. of Mathematics*, vol. 29 (1907), p. 4.

† *Transactions Amer. Math. Society*, vol. 4 (1903), p. 153.

‡ Vol. 34 (1905), p. 160.

The group of isomorphisms of every finite group is finite since its order cannot exceed $(g-1)!$, g being the order of the group. There are only four groups for which the order of I has this maximal value; viz., the groups of orders, 1, 2, 3 and the non-cyclic group of order 4. As a rule the order of I is very much smaller, since the possible holomorphisms are greatly restricted by the properties of the operators. It has been observed that in a non-abelian group not more than three-fourths of the operators may correspond to their inverses and the groups which have this property have been considered. From this fact it follows that an abelian group may be defined by the property that more than three-fourths of its operators may correspond to their inverses in a holomorphism of the group.[‡] Manning has considered the groups in which five-eighths or more of the operators may correspond to their inverses and in this connection proved a fundamental theorem in regard to the properties of groups involving two invariant subgroups which have only identity in common.[§]

The group of isomorphisms of the cyclic group of order $2 \cdot 3^m$ is the cyclic group of order $2 \cdot 3^{m-1}$. Hence it follows that each of the m successive groups of isomorphisms of this cyclic group as well as of the cyclic group of order 3^m is cyclic while the $(m+1)$th is identity. The only other groups which have the property that all their successive groups of isomorphisms are cyclic are the group of order 4 and those whose order is a prime number of the form $2 \cdot 3^m + 1$. From the fact that the order of the group of isomorphisms of every group whose order exceeds 2 is greater than unity it follows that we arrive at an infinite system of groups by forming the successive holomorphs of any group whose order exceeds 2. The characteristic operators of a group are the invariant operators of its holomorph. Since *an abelian group cannot have more than one characteristic operator besides identity,* it results that each of the successive holomorphs of an abelian group has either one or no invariant operator besides identity, as the abelian group has one or no characteristic operator in addition to identity. The question whether a non-abelian group can have more than one characteristic operator besides identity remains unsettled. This is also true of the question whether a non-abelian group can have an abelian group of isomorphisms. If such a group exists there must be a metabelian group of order p^a which has the same property.

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[*] Young, *Amer. Jour. of Mathematics*, vol. 25 (1903), p. 206.

[†] Ranum, *Transactions Amer. Math. Society*, vol. 8 (1907), p. 89.

[‡] *Annals of Mathematics*, vol. 7 (1906), p. 59.

[§] Manning, *Transactions Amer. Math. Society*, vol. 7 (1906), p. 223.

The characteristic properties of a complete group are that it does not admit outer isomorphisms and none of its operators except identity is invariant under the group. A large number of well-known groups have the latter property without having also the former. It seems desirable to find groups which have the former property without having also the latter. In fact, the second part of the definition can only be justified by a proof of the existence of such a group. This proof is included in the proof that there exists a group of composite order which is both simple and complete since the direct product of such a group and the group of order 2 admits no outer isomorphisms but includes an invariant operator of order 2.* In view of the historic interest in the five-fold transitive groups of degrees 12 and 24 respectively it is of special interest to note that the latter is a complete group while the former has a group of twice its own order for its group of isomorphisms.

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* *Messenger of Mathematics,* vol. 37 (1907), p. 54.

§ 4. SUBSTITUTION GROUPS.

In 1896 "Le grand prix des sciences mathématiques" was awarded to Maillet for his memoir entitled "Recherches sur la classe et l'ordre des groupes de substitutions." The memoir was not published until about six years afterwards * and hence its publication comes within the period covered by the present report. The memoir covers 120 pages and is divided into two parts. The first is devoted to the class of the primitive substitution groups which are simply isomorphic with either a symmetric or an alternating group, and it is followed by two notes in which hypersystems and primitive groups whose class is not less than four-fifth of the degree are considered. The second part, which the author considers the more important, is devoted to the limit of the order of the groups of degree n which do not involve the alternating group of this degree.

The term hypersystems is introduced with a view to exhibiting more clearly the connection between substitution groups and Lie's transformation groups, as well as to generalize some results previously published. Most of the results are extensions of theorems due to Jordan and Bochert. Some of these relate to the class of a primitive group which does not include the alternating group of the same degree. Manning considers the same question in several recent memoirs. † In most of the older work along this line the degree of such a group was assumed to be given and the smallest possible class of the primitive group was considered, while Manning assumes the minimum class and finds the maximum degree for this class.

The important theorem that every substitution group of prime degree p which contains more than one subgroup of order p is at least doubly transitive was first proved by means of group characteristics. Burnside has recently given a much simpler proof based upon a purely arithmetical property of the prime roots of unity. ‡ The enumeration of all the possible substitution groups of a given degree has been materially advanced during the period under consideration. Miss Martin's enumeration of the imprimitive groups of degree 15 was completed by the list published by Kuhn. This list is preceded

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* Maillet, *Mémoires présentés par divers Savants a l'Académie des Sciences de l'Institut national de France*, vol. 32 (1902).

† Manning, *Transactions Amer. Math. Society*, vol. 4 (1903), p. 351 ; vol. 6 (1905), p. 42 ; BULLETIN, vol. 13 (1906), p. 20.

‡ Burnside, *Quar. Jour. of Mathematics*, vol. 37 (1906), p. 215.

by a number of new theorems relating to the construction of such groups and groups which involve all the substitutions which are commutative with each of the substitutions of a transitive group.*

The investigations of Sylow in regard to groups of a prime degree which were noted in the preceding report have been extended by Frobenius, who proved that there are only four groups of degree p which contain exactly $p + 1$ subgroups of order p. This theorem was also proved by de Séguier without the use of group characters in a memoir which, like the one by Frobenius, contains a large number of other new theorems on substitution groups.† From the results of the last two paragraphs it follows that each of the unknown transitive groups of degree p may be assumed both to be multiply transitive and also to involve more than $p + 1$ subgroups of order p. Hence these groups contain transitive subgroups of lower degree and therefore belong to a category which has recently been studied by Manning.‡ The special case where $p = 2q + 1$, q being a prime, was considered by de Séguier in the memoirs just noted and he arrived at the result that there is no transitive group ot degree 23 in addition to the 7 which are well known. This result has been verified by the writer. It may be observed that the smallest possible number of transitive groups of degree $p > 5$ is 6. Jordan proved that this is also the actual number when p is either 47 or 59. Rietz proved that for every value of $p > 11$ of the given form there exists a simple group of composite order which can be represented as a simply transitive primitive group of degree $1 + kp$.§

A fundamental theorem relating to any transitive substitution group G of degree n has been stated as follows: If p^β is the highest power of p which divides n, each Sylow subgroup of order p^α in G has a transitive constitutent of degree p^β and all its other transitive constituents are of degree of $p^{\beta+\gamma}$ ($\gamma \geqq 0$). If $n = 2q^{\alpha_1}$, q being any odd prime, each of the Sylow subgroups whose order is a power of q has just two transitive constituents of degree q^{α_1}. If n is a power of a prime, a Sylow subgroup of G whose order is a power of the same prime is transitive.[*]

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* Kuhn, *Amer. Jour. of Mathematics*, vol. 26 (1904), p. 45.

† Frobenius, *Berliner Sitzungsberichte*, 1902, p. 351; de Séguier, *Comptes rendus*, vol. 137 (1901), p. 37, and *Liouville*, vol. 8 (1902), p. 253.

‡ *Transactions Amer. Math. Society*, vol. 7 (1906), p. 497.

§ Rietz, BULLETIN, vol. 11 (1905), p. 544.

The important applications of group characters to such problems as the determination of the least number of variables by means of which a given abstract group may be represented as a linear group, and of all the linear homogeneous substitution groups which are either simply or multiply isomorphic with a given group, have continued to attract many investigators to this field with a view either to extensions of the theory or to making it more accessible. Dickson, Burnside, and Schur have published valuable expository articles on the theory of group characters in which the subject is approached from different standpoints.[†] While Burnside makes considerable use of Hermitian forms in his fundamental theorems, Schur replaces these by simpler considerations so that he presupposes only an elementary knowledge of matrices from the theory of linear substitutions. Both of these authors employ methods which differ widely from those used by Frobenius, and Schur develops his theory under the heading "New foundation for the theory of group characters."

Frobenius proved various theorems relating to the group characters of the multiply transitive groups which do not involve the alternating group of the same degree and gave a number of historical data relating to these groups. Among these theorems are the following: A character of the symmetric group whose dimension does not exceed $\frac{1}{2}r$ is also a character of every r-fold transitive group of the same degree. The necessary and sufficient condition that a group is either two-fold or four-fold transitive is that it has the character $\alpha - 1$, or the characters $\alpha - 1$, $\frac{1}{2}\alpha(\alpha - 3) + \beta$, $\frac{1}{2}(\alpha - 1)(\alpha - 2) - \beta$ respectively, where $\alpha, \beta, \gamma, \cdots$ are the number of cycles of degrees $1, 2, 3, \cdots$ in a substitution.[‡] By means of a fundamental theorem relating to the roots of unity Burnside establishes several theorems which are useful in calculating group characters, and by way of illustration he determines the characters of the simple group of order 504.[§] Alasia[‖] has also given a brief exposition of the characters of several groups.

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[*] BULLETIN, vol. 9 (1903), p. 543.

[†] Dickson *Ann. of Math.*, vol. 4 (1902), p. 25; Burnside, *Proc. London Math. Society*, vol. 1 (1903), p. 117; Schur, *Berliner Sitzungsberichte*, 1905, p. 406.

[‡] Frobenius, *Berliner Sitzungsberichte*, 1904, p. 558.

[§] Burnside, *Proc. London Math. Society*, vol. 1 (1903), p. 112.

[‖] Alasia, *Rivista di Fisica, Matematica e Scienze naturali*, anno. 6 (1905), p. 1905.

Loewy published several important memoirs which are principally devoted to the study of the reducibility of the linear homogeneous substitution groups. Most of his theorems apply to groups of infinite order but he has also considered groups of finite order.[*] One of his theorems was extended by Dickson, who established its validity for much more extensive domains.[†] All the finite collineation groups in three variables have recently been determined by Blichfeldt, who employs more direct methods than those used by Jordan and Valentiner in their much earlier investigation of this problem. Blichfeldt gives also an enumeration of the principal imprimitive collineation groups in four variables, together with their generating substitutions, and a complete list of the primitive ones based upon various theorems which were developed by him principally in the *Transactions* of this Society.[‡]

Schur has given a complete solution of the problem of finding all the possible representations of a finite group as a linear fractional substitution group, and, together with Frobenius, he established the following fundamental theorem: A finite group of linear substitutions is equivalent to a real group when its substitutions transform into itself a quadratic form of a non-vanishing determinant, and only then; two isomorphic groups of linear substitutions have the same irreducible components if any two corresponding substitutions have the same trace (Spur) and only then.[§] The fundamental theorem proved by Jordan about thirty years ago, which establishes the fact that every finite linear homogeneous group on n variables contains an invariant abelian subgroup such that the order λ of the corresponding quotient group is less than a certain number depending only upon n, has been extended both by Blichfeldt and by Schur. The former has found further restrictions for λ in the article published in the *Transactions* to which we have just referred, as well as in an earlier article in the same journal, while the latter has considered the possible orders of the linear groups when the trace of each substitution is restricted to certain fields.

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[*] Loewy, *Transactions Amer. Math. Society*, vol. 4 (1903), pp. 44 and 171; ibid., vol. 6 (1906), p. 504; Verhandlungen des dritten internationalen Mathematiker-Kongresses (1904), p. 194.

[†] Dickson, *Transactions Amer. Math. Society*, vol. 4 (1903), p. 434.

[‡] Blichfeldt, *Mathematische Annalen*, vol. 63 (1907), p. 552; ibid., vol. 60 (1905), p. 204; *Transactions Amer. Math. Society*, vol. 6 (1905), p. 230.

[§] Frobenius and Schur, *Berliner Sitzungsberichte* (1906), p. 186; *Crelle*, vol. 127 (1904), p. 10.

In this case it is possible to find a number which is divisible by every possible group and which depends only upon the field and the number of variables.*

Shaw has investigated the algebras defined by finite groups. "By a group algebra is meant that linear algebra whose units are defined to be such that each unit e_i corresponds to an operator O_i of some given finite abstract group, and conversely, and such that to each equation of the group $O_iO_j = O_k$ corresponds an equation $e_ie_j = e_k$ of the algebra." In this investigation Shaw confines himself to the scalar continuous field and observes that if the coefficients are any numbers in such a field, every abelian group of the same order gives the same group algebra, a result which is not true for all fields of coefficients. He points out the ultimate connection between group algebras and the theory of group characters and group determinants, and cites Poincaré's fundamental theorems bearing on such algebras.†

In volume 23 of the *Mathematische und Naturwissenschaftliche Berichte aus Ungarn*, Visnya determines a necessary and sufficient condition that a finite group of linear substitutions is intransitive and he also considers all the possible Hermitian invariants of such a group. A sufficient condition for its intransitivity is due to Maschke and was published much earlier. Burnside has recently given some new criteria for the finiteness of the order of a group of linear substitutions. The coefficients in the substitutions of a group of homogeneous linear substitutions are generally complex numbers of the form $\alpha + \beta i$, where α and β are real numbers. If the group is of finite order, there is a finite number of coefficients and therefore there is a finite positive number M such that for each coefficient $|\alpha| < M$ and $|\beta| < M$. Similarly there must be another positive number m such that for each coefficient $|\alpha| = 0$ or $> m$ and $|\beta| = 0$ or $> m$. The existence of these two numbers is proved to be both a necessary and a sufficient condition that the group is finite. Another necessary and sufficient condition is expressed by the following theorem: "If, in a group of linear substitutions on a finite number n of symbols, the order of every substitution is equal to or less than a finite number m, then the group is of finite order." Burnside proves this theorem* without giving reference to the less general theorem proved by Loewy which is mentioned at the end of my report on the groups of an infinite order.† Another test given in this paper for the finiteness of a linear substitution group on a finite number of symbols is that it contains a finite number of distinct sets of conjugate substitutions.

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* Schur, *Berliner Sitzungsberichte* (1905), p. 1.

† Shaw, *Transactions Amer. Math. Society*, vol. 5 (1904), p. 326.

Quite recently Loewy has investigated the groups of linear homogeneous substitutions which are of the type of a finite group and gave a complete development of their theory.‡ Dickson has recently published two papers in which he considers for the first time the problem of representing a given finite group as a linear congruence group.§ He points out that the only one of the different expositions of Frobenius's theory of group characters mentioned above which may be utilized in the construction of a corresponding modular theory is that by Schur. While the developments of Frobenius relate to the representation of a given finite group as a non-modular linear group the work of Dickson employs a modulus in such a representation.

UNIVERSITY OF ILLINOIS,
July, 1907.

THE DRESDEN MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

THE 1907 meeting of the Deutsche Mathematiker-Vereinigung was held in Dresden, September 15–21, in conjunction with the 79th convention of the Naturforscher und Aerzte. The meeting took place in room 80 of the Technische Hochschule. In commemoration of the 200th anniversary of Euler's birth a considerable number of the papers were devoted to an exposition of his services to science. The following papers were read:

1. K. ROHN, Leipzig: "Algebraic space curves" (report).
2. F. KLEIN, Göttingen: "Concerning the connection between the so-called theorem of oscillation of differential equations and the fundamental theorem of automorphic functions."

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* Burnside, *Proc. London Math. Society*, vol. 3 (1905), p. 435.

† BULLETIN, vol. 7 (1900), p. 121.

‡ Loewy, *Mathematische Annalen*, vol. 64 (1907), p. 264.

§ Dickson, *Transactions Amer. Math. Society*, vol. 8 (1907), p. 389.

3. G. LANDSBERG, Kiel: "Theory of curvature and calculus of variations."

4. A. v. BRILL, Tübingen: "Introduction to the Euler commemoration."

5. L. SCHLESINGER, Klausenburg: "On a problem of diophantine analysis, as considered by Fermat, Euler, Jacobi, and Poincaré."

6. A. PRINGSHEIM, Munich: "Concerning Euler's transformation of series."

7. E. BRAUER, Karlsruhe: "Euler's theory of the turbine."

8. F. S. ARCHENHOLD, Treptow: "Some letters of Euler."

9. R. GANS, Tübingen: "Euler as physicist."

10. H. E. TIMERDING, Strassburg: "Euler's investigations in nautical mechanics."

11. W. HORT, Gross Lichterfelde: "Euler's significance in technical science."

12. E. HOPPE, Hamburg: "Euler's place in the theory of optics."

13. L. SCHLESINGER, Klausenburg: "Development of the analytic theory of linear differential equations since 1863" (report).

14. A. SCHOENFLIES, Königsberg: "On the so-called Richard paradox in the theory of point sets."

15. F. HAUSDORFF, Leipzig: "On dense types of order."

16. H. WIENER, Darmstadt: "Geometric theory of invariants of binary forms."

17. V. Varicak, Ogram: "Contributions to non-euclidean geometry."

1. Professor Rohn's report will soon appear in the Encyklopädie der mathematischen Wissenschaften, III, C. 8.

4. In a short paper Professor v. Brill called attention to Euler's birthplace, briefly sketched his life, expressed the belief that Germany's celebration in Euler's honor would find comprehending appreciation throughout the civilized world, and discussed its significance for the history of science.

7. Professor Brauer's paper will appear in the *Jahresbericht.*

9. Professor Gans gave the following historical summary: Euler published important memoirs in nearly all fields of physics. Above all, he created analytic mechanics, which, in

contrast with the earlier geometric method, supplies an ever-ready recipe for new as well as old problems. Before Euler's time even geniuses contented themselves with easy problems; since then, anyone can calculate by rule the path of a given mass-point under the action of determinate forces. Applying his theory to astronomy, Euler calculated by the method of variation of the constants, the perturbative effect of celestial bodies upon the orbit of a planet. He rendered assistance to the science of artillery by his approximative treatment of the path of a projectile; to nautical science by his theory of the ebb and flow of tides and of the building and sailing of ships. He investigated the bending and elastic swing of beams and calculated the safety load of a column, thus rendering service to technical science. He studied the theory of sound and investigated the physical basis of musical consonance and dissonance. The results of his work on optical instruments, which fill three thick volumes, have contributed much to the technique of microscope and telescope making, and, above all, have resolved the physics of catoptrics to that of dioptrics. His discovery of the achromatic telescope deserves especial mention, since it was at Euler's instigation that Dollond made his glass. Euler's ether theory is of great importance for theoretical physics and natural philosophy. He assumes the universe filled with a fine substance to whose vibrations the phenomena of light are due and whose currents are the ultimate cause of gravitation, electricity, and magnetism. It is not to be wondered at that this ether theory seems imperfect from our present standpoint, for in Euler's time Coulomb's law and the connection between light, electricity, and magnetism were unknown. Besides physical proofs Euler sometimes used theologic-teleological proofs, although his mathematical-analytic interest is frequently in the foreground. But in cases of application to astronomy or technical science no method is too uninteresting for him. We admire in him the powerful physical analyst, who studied the details, and the great natural philosopher, who sought the unities of natural phenomena.

10. Professor Timerding discussed those works of Euler which treat of naval mechanics. He indicated the principal points of view from which these works, especially the Scientia navalis, 1749, and the Théorie complète de la construction et de la manoeuvre des vaisseaux 1773, may be judged. Two things must be con-

sidered — personal relation to Euler's scientific temper and to his other investigations, and technical relation to other investigations concerning naval mechanics and to the development of rational naval construction. One should emphasize, for example, the relation to Euler's discovery of the laws of motion of rigid bodies and, on the other hand, the impulse to a critical investigation of the strain of the ship by reason of the coming into play of different forces. It should also be mentioned that Euler knew all three kinds of practicable propellers and had calculated their efficiency. The author emphasized the significance of Euler's systematic development for the subsequent progress of the theory of the ship.

No. 11 is to be published at once in the *Physikalische Zeitschrift.*

12. Professor Hoppe drew attention to the fact that Euler was the first to treat the vibrations of light analytically and to deduce the equation of the curve of vibration as dependent upon elasticity and density. Euler distinguished simple rays of homogeneous wave length from those of white light, calling them elementary rays generating elementary colors. He deduced the law of refraction analytically and explained that the rays of greater wave length must suffer the least deviation. He also investigated non-transparent bodies with regard to their behavior toward light. He studied dispersion in the search for a corrective for chromatic aberration, which Newton had declared unattainable. For this purpose he found the formula that the ratio of the natural logarithms of the indices of refraction must remain constant for different colors. If the ratio is α then the dispersion is given by $dn = (1 - \alpha)(1 + \alpha)^{-1} nln$. It was this investigation that induced Dollond to construct his achromatic lenses. Euler calculated a large number of composite telescopes and microscopes, thereby increasing the possibilities in the construction of these instruments. The results of his investigation with the composite ocular are of special importance. Euler treated atmospheric refraction of light analytically both for celestial and terrestrial objects, finding for the latter an expression that is fundamental in leveling. With regard to photometry Euler distinguished between intensity of light and of illumination and discovered the dependence of the latter upon the inclination of the surface and the decrease of the

former with the distance. He also called attention to the optical methods for eliminating the third dimension. Euler was thus the only physicist of the eighteenth century who advanced the undulatory theory.

14. Professor Schoenflies exposed the fallacy in the reasoning of Richard and showed that a parallel line of argument could be made to invalidate the results of all human thought.

15. Denoting by *series* (Reihen) sets of the type of Cantor's "Zahlclassen," or of the inverse type, one can assign a definite character to each element and to each gap of a dense set, according to the series which the element, or the gap, bounds to right or left. Upon this Professor Hausdorff based a classification of dense sets and an existence proof for the a priori obvious kinds. In the first case, in which the dense set contains series of potency one, there are two different species, represented by the linear continuum and by the set of rational numbers. In the case of series of potency two, there exist 210; of potency three, 243376 different species.

17. Professor Varicak's paper is in abstract as follows. Erect two normals to the x-axis at points $N_1(n_1, 0)$, $N_2(n_2, 0)$. There will be four straight lines gi parallel to these normals, uniquely determined by the intercepts n_1, n_2 and by the sense in which the gi shall be parallel. Making use of Lobachevsky's coordinates and assuming $\epsilon_1(\epsilon_2) = +1$ when the first (second) end of gi lies in the positive half plane, but equal to -1 in the contrary case, we can write the equation of these four parallels as follows:

$$e^x + \epsilon_1\epsilon_2 e^{n_1+n_2} e^{-x} - (\epsilon_1 e^{n_1} + \epsilon_2 e^{n_2}) \operatorname{sh} y = 0.$$

This is the general equation of the straight line which can often be used to advantage. In the second part of his paper the author characterized some movements of the Lobachevsky plane by means of infinitesimal transformations. Translations along the "Abstandslinie," the normal and the parallel to the x-axis respectively are given by

$$Uf \equiv \frac{1}{\operatorname{ch} b} \cdot \frac{\partial f}{\partial x}, \quad Uf \equiv \frac{\partial f}{\partial y}, \quad Uf \equiv (1 - e^{-2x} \operatorname{sh}^2 b) \frac{\partial f}{\partial x} - \operatorname{sh} y \frac{\partial f}{\partial y}.$$

Rotations about the positive end of the x-axis and about the origin are given respectively by

$$Uf \equiv \frac{\sqrt{e^{2(x-b)}-1}}{e^{2(x-b)}} \cdot \frac{\partial f}{\partial x} + \frac{1}{\operatorname{ch} y} \cdot \frac{\partial f}{\partial y},$$

$$Uf \equiv -\frac{\operatorname{sh} y \operatorname{ch} x}{\operatorname{sh} p} \cdot \frac{\partial f}{\partial x} + \frac{\operatorname{sh} x}{\operatorname{sh} p} \cdot \frac{\partial f}{\partial y}.$$

Invariant pencils of curves can be determined easily by Lie's method.

At the business meeting of the Vereinigung a committee, consisting of Professors Pringsheim, Stäckel, and Krazer, was appointed to consider ways and means looking toward the publication of Euler's works. It was hoped that a report of progress could be made at the meeting in Rome next April. Professor Klein was elected to succeed Professor Pringsheim in the Vorstand and will be chairman for the current year. Professor Krazer was continued as Secretary.

During their stay in Dresden the visiting scientists did not lack for entertainment. The evenings were socially filled as follows: Sunday, September 15th, there was an informal initial gathering in the main hall of the permanent exposition building for the purpose of renewing old acquaintanceships and forming new ones. On Monday evening the Belvedere was elaborately decorated with colored lights in honor of the visitors. By reason of inclement weather the open air concert had to be abandoned, however, in favor of indoor entertainment. Tuesday evening the Naturforscher were the guests of the King of Saxony at a superb presentation of La Bohême at the royal opera house. Wednesday evening the visitors banqueted themselves, Thursday evening they were received at supper by the mayor of Dresden, in the main hall of the exposition building. During the entire week the city of Dresden, with its wealth of attractions, was hospitably open to the visitors, while abundant opportunity was offered to those who desired to make excursions, at reduced rates, to neighboring points of interest.

C. A. NOBLE.

GENERAL THERMODYNAMICS.

Thermodynamics: An Introductory Treatise Dealing Mainly with First Principles and their Direct Applications. By G. H. BRYAN. Teubner's Sammlung, Band XXI. Leipzig, 1907. xiv + 204 pp.

The books on thermodynamics are legion. In purpose and content they run the whole gamut from Maxwell's inimitable little work on heat and the many no more advanced though greatly inferior treatments of the subject to the large treatises like Weinstein's Thermodynamik und Kinetik der Körper in three sizable and highly mathematical volumes of which the last is not yet finished. Some are chiefly concerned with engineering problems, others with physics, and lately many with physical chemistry. One might therefore well ask what was left for a new work on thermodynamics except repetition. Anybody who has undertaken to teach a comprehensive course on thermodynamics, beginning essentially at the foundations and covering as well as may be the various applications up to and even including those in physical chemistry, will have no hesitation in answering that there is no short treatise which he could use as a guide — as a précis — for his whole course, nothing that at once was comprehensive and compact, as deductive as might be and yet thoroughly physical, and furthermore well balanced between the distracting needs of the different dependent fields of science. Better might it be said that such would have been the answer a few months ago. Now it would no longer be true; for Bryan's little book furnishes precisely this desideratum.

It should be noted that there is no necessity of banishing the physical side of a science in giving a deductive presentation of its theory. The history of most branches of mathematical physics is about this: First some fundamental experiments combined with a very crude theory which is frequently almost as much wrong as right; then after an increasing accumulation of facts there appears some master mind to formulate a comprehensive and essentially correct theory based upon a few crucial experiments; finally the deductive stage arrives in which the start is made from certain mathematical statements and physical experiment no longer plays any considerable rôle. This last stage is, however, that in which physical *interpretation* should be most in evidence, and it is the omission of this constant a posteriori contact with nature that renders so much mathematical physics futile as far as physical science is concerned. The advance must not be away from physics and into mathematics; it should be from a priori physics and a posteriori mathematics to a priori mathematics and a posteriori physics. This last point of view is apparently that adopted by Bryan in regard to thermodynamics, as it is by most modern authors in regard to mechanics and the classical theories of optics.

The author divides his work into three parts: Physical aspects of thermodynamics, pages 1–26. The foundations of rational thermodynamics, pages 27–96. Thermodynamics of particular systems, pages 97–198.

From the title of the first part it might appear that we were still in the stage of a priori physics and close contact with experiment. A perusal of the text will rapidly disabuse one of any such notion. As a matter of fact only such reference to material bodies is made as is necessary for the formulation of the definitions and laws upon which thermodynamics is to be built. It would not do to say that in this first part the author lays down axioms for a thermodynamics; his procedure, subject to later revision, is analogous to Newton's in mechanics. Heat is taken as a fundamental idea just as force was. In fact as we find in mechanics the law that, when force acts, the amount of momentum is changed by the amount of the impulse of the force, so we find here the law that, when heat is transformed into work or conversely, the quantity of heat gained or lost is proportional to the quantity of work lost or gained. And there is the immediate additional statement that this law affords a measure of heat just as Newton's law gives a measure of force. Some might wish to banish heat from thermodynamics as they have banished force from dynamics. The attempt to do so does not seem particularly valuable from a pedagogic point of view, and it is probably well that Bryan is content at this early stage of his work to stay his desire for deduction just where he does. The concluding chapter of part I deals with the necessary changes of variable incident to the transformation of the many thermal coefficients previously defined.

Part II begins with an admirable discussion of various behaviors of energy — conservation, localization, reversibility, availability, and the dependence of the last on external conditions.* This might all find its way quite appropriately into many treatises on mechanics. The author then proceeds to discuss irreversibility with the definition that a change is said to be irreversible when the system cannot of itself change back to its original condition, and he lays down two axioms (from now on the word axiom is often in evidence): 1) If a system can undergo an irreversible change it will do so. 2) A perfectly reversible change cannot take place of itself; such a change can only be regarded as the limiting form of an irreversible change. It seems as though the second axiom more or less vitiates the immediately preceding definition of irreversibility. Although the difficulty is obviated before many pages have been turned, it might have been better to phrase the statements a little more carefully. The same may be said of the later sentence which expresses the fundamental principle of degradation: In all irreversible transformations available energy is lost, never gained. If anything in the way of emphasis is to be added to lost, it would be well to say something as definite as never gained or conserved. Lost and gained do not together form a universe of discourse in this case. These, however, are small matters to throw in the face of an author who is perhaps the first to have thoroughly conscientious scruples as regards definitions and axioms in his upbuilding of thermodynamics.†

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* The author's characterization of energy on page 39 is far from possessing the completeness and definiteness which is desirable in an entity that is to be made the basis of his whole deductive system. To critics inclined to be too severe at this point, may be replied with great reason that in the present state of our knowledge it is apparently impracticable to build up a perfectly definite system of rational thermodynamics. Indefiniteness is bound to occur somewhere unless the number of undefined elements is made larger than Bryan wishes to make it. To leave energy a somewhat vague and elastic conception is probably far better than to have some other entity obscure or undefined — and for precisely the reason that energy is so familiar and fundamental in all physics as to spare definition better than anything else. Furthermore, from a logical point of view, it should be noted that with the statements on page 39 must be included all the later axioms which go to contribute additional precision to the basic undefined symbol "energy."

† It is only fair to mention at this point Perrin's Traité de chimique physique: les principes, Paris, Gauthier-Villars, 1903. Perrin is here writing for chemists. He goes so carefully into all the fundamental ideas of thermodynamics as to make his book almost tedious reading for anybody who knows as much about energy and entropy as a chemist ought to; but as the chemists do not know as much about such things as they should, the author is doubtless forced to follow the laborious detail that he does. Furthermore Perrin chooses to keep in much closer contact with experiment than Bryan and to

After some further treatment of the characteristics of a system we call thermodynamic, there appears the definition of the quantity of heat involved in a thermodynamic transformation, namely,

$$dQ = dU + dW,$$

where dU is the increase of energy in the system and dW the amount of external work done by the system. A discussion of the Carnot's cycle in its relation to the principle of the degradation of energy, which the author adopts as more fundamental than the usual statement of the second law, then leads to the definition of temperature and of absolute temperature (thermodynamic scale) in particular. The author's presentation of this whole introductory matter is certainly not hackneyed. It can only be characterized as original. It is extremely illuminating and peculiarly satisfactory to the mathematician who naturally demands closer formal reasoning than the physicist. It is to be hoped that future treatments of the subject will borrow much of the clearness and logic of Bryan's even in such frequent cases as those in which a complete adoption of his method would be inadvisable for one reason or another.

Thus far we have covered six chapters. The remaining chapters of this part may be spoken of more briefly. They deal with the usual topics. Chapter VII takes up the relation of available energy to entropy. Two definitions of entropy are given and each is elaborated. Homogeneous and heterogeneous systems are alike treated in detail, and so with both reversible and irreversible transformations. This chapter is long and clear; every reader has ample opportunity to become so thoroughly familiar with entropy as to forget that it ever was considered a weird bugaboo. The following two chapters deal with the matters of stability and thermodynamic potentials.

Part III is on the thermodynamics of particular systems. That the author should start off with the treatment of radiation is a bit of brave novelty. As a matter of fact such systems

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depend less on rational dynamics; his work is not so completely deductive. This again is probably the wisest course for him to follow. To any who find difficulty in Bryan's Thermodynamics, especially in these underlying chapters, Perrin's Principes is heartily to be recommended; and even for those who find no trouble in following Bryan, Perrin has much of interest to offer.

are perhaps the simplest, and the deduction of Stefan's law with the further proof that optical methods cannot increase availability gives to the subject at the very start an interest and vigor which the older treatments of perfect gases cannot approach. The succeeding chapter is largely a collection of formulas expressing the various thermal coefficients of simple systems in terms of the potentials. Then follows in chapter XII some account of gaseous systems whether pure or mixed. No very great detail is entered upon; true to the author's general intention the chapter is merely a sketch. Chapter XIII contains an account of the determination of thermodynamical quantities and will be especially welcome to those who like to see how the fundamental magnitudes are derivable from experimental data.

The next three chapters, XIV, XV, XVI, touch on the physico-chemical aspect of thermodynamics. The first takes up van der Waals's equation and the critical states, and also the phase rule and triple point for a single substance. The second treats the general question of phase equilibrium for heterogeneous substances. The third, which deals with reversible thermoelectric effects, gives Helmholtz's and Gibbs's formulas for reversible elements and an excellent discussion of the Kelvin and Peltier effects. In the next and last chapter are found some geometric and dynamical representations of thermodynamic phenomena. As might be expected these are for the most part Gibbs's surfaces and related diagrams on the one hand and the cyclic dynamical systems on the other. It seems as if it might have been well to add just a few words more on strictly statistical methods with perhaps a mention of the analogies to thermodynamics which Gibbs pointed out in his last work.* It should be borne in mind, however, that to say merely a few words which shall be comprehensible is a very difficult matter in a subject so technical and formulaic as statistical mechanics.

There remains merely a short conclusion, which again and for the last time emphasizes the fundamental points of the text. An elaborate index closes the volume. Although this is scarcely necessary in a work so concise and so well arranged as the present, it is surely not in the way.

From the foregoing sketch of the contents and general aim of Bryan's two hundred pages, it may be possible to gain some insight into the character of the work, and to see that an empty place in our scientific literature has been discovered and satisfactorily filled. As was stated before, this book furnishes an excellent précis for anybody lecturing on general thermodynamics. If in addition there were in existence an elaborate treatise of three or four times the length following the same order of arrangement and amply supplied with physical applications and developments, the student of thermodynamics would scarcely have need of any lectures. As it is, both student and teacher are so much better off than they were a twelvemonth ago that it would be a bit ungracious already to ask for more —or would it be the highest expression of gratitude?

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* Elementary Principles in Statistical Mechanics, Charles Scribner's Sons, New York, 1907.

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, *September*, 1907.

SHORTER NOTICES.

Funzioni poliedriche e modulari. By G. VIVANTI. Milan, Hoepli, 1906. viii + 437 pp.

THE object of this work cannot better be characterized than by the following quotation from the preface:

"The present volume has the modest scope of preparing the reader for the study of the classic lectures of Klein on the icosahedron and of the treatise on the elliptic modular functions by Klein and Fricke. The first of these, a model of elegance and a veritable mine of new and interesting ideas, is quite difficult to read both on account of the many concepts which are barely outlined and still more because, even after the various details have been understood, the connection which binds them, the guiding idea, is far from evident and comes to light only after a profound study and a thorough review of the entire material. The second work, on account of the mass and the multiformity of its contents, does not lend itself readily to an introductory study. On account of the lack of a suitable treatise for facilitating acquaintance with the theory of the polyhedral and modular functions, I believed it my duty to make available for others the work of elaboration which I have performed for my own use, thus saving them the repetition of this useful but laborious work.

"It may be too bold an attempt to include in the brief space of a small manual the principles of two vast and important theories. Nevertheless it seemed useful, in spite of the greater conciseness required, to unite both theories in order to avoid the repetition necessary in treating them separately, and also to show that the icosahedron theory, which is capable of appearing in the field of analysis as an elegant and independent creation of a special type, is only the first of a series of related constructions very closely bound together. From this point of view it would have been better still to take another step and include also the theory of the automorphic functions; but this would clearly be impossible.

"To save space, and for greater symmetry of treatment, I have assumed that the reader, besides having a certain familiarity with the more elementary parts of mathematics, has also some notions of several more advanced theories: analysis situs, functions of a complex variable and Riemann surfaces, elliptic functions, abelian integrals, linear differential equations, theory of numbers."

The program which the author thus places before himself has been carried out, we think, with much care and good judgment. He has treated his subject with admirable simplicity, directness, and unity. Not only will this work greatly facilitate the efforts of the reader to master these comprehensive theories, but it will be of particular service to those whose main interests lie outside this special field, by enabling them to become familiar with its general topography without an undue expenditure of time.

The book is printed in large, well leaded type. By using a paper as thin as is consistent with opaqueness and by cutting down to a narrow margin, the publisher has produced a small and handy volume without any sacrifice of legibility.

J. I. HUTCHINSON.

Vorlesungen über Zahlentheorie. Einführung in die Theorie der algebraischen Zahlkörper. By J. SOMMER. Leipzig, B. G. Teubner, 1907. iv + 361 pp.

THE generalizations of the ordinary theory of numbers which, following Gauss's introduction of complex integers, have been made by Kummer, Dirichlet, Dedekind, and Kronecker constitute an extensive and exceedingly interesting part of mathematics.* Nevertheless it is true that this subject is not as widely read as is proper in view of its importance; and undoubtedly the reason is to be found in its abstract nature.

The recent books of Bachmann † and König ‡ upon this general subject are excellent, but many will find them very difficult because of the generality of their treatments. Indeed, Professor Dickson in reviewing them § felt it necessary to give a brief exposition of the theory for a special case, and there is no doubt of the desirability of an extensive treatment of the simpler cases which will lead up to the general discussion. The volume under review has been prepared to serve those who have hitherto been hampered by the lack of an introduction to the subject.

The author has successfully undertaken to present the chief points of the theory by means of the elementary cases of (absolute) quadratic and cubic number·fields and fields quadratic with respect to a quadratic fundamental field. The book is divided into five chapters, the first of which constitutes an exceedingly brief introduction, so brief in fact that it must be regarded as simply a statement of those topics of the ordinary theory of numbers whose generalizations are to be given later and are to form the foundation of the discussion. And, naturally enough, these topics are the divisibility of integers, the function $\phi(n)$, congruences, and Fermat's theorem.

The second and third chapters constitute two thirds of the book and are the most important ones. In the second the theory of quadratic number fields is worked out in detail in connection with the topics above mentioned, and in this chapter the reader is given an opportunity to become intimately acquainted with the ideas of number field, ideals, and the generalizations of the results obtained by Fermat and Gauss. In chapter three applications of the theory of quadratic fields are made to several important problems, and particularly welcome to many will be the introduction of the famous "last theorem" of Fermat.

To give the reader a closer approach to the generalities of the subject, the author devotes a fourth chapter to such a broadening of methods as will make the treatment adequate for the case of cubic number fields. A fifth chapter deals with fields which are quadratic with respect to another one, itself a quadratic field. Both of these chapters are relatively short and are not intended to be exhaustive.

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* See Hilbert, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 4.

† Zahlentheorie, Teil 5, 1905.

‡ Einleitung in die allgemeine Theorie der algebraischen Grössen, 1903.

§ BULLETIN, April, 1907.

The value of the volume is distinctly increased by the introduction of a considerable number of simple illustrations of the theory in addition to the more elaborate applications made in the third chapter.

The book is timely, has been put out by the publishers in an attractive form, and should materially increase the number of those who will undertake to familiarize themselves with this branch of general algebraic theory.

GEORGE H. LING.

Les Principes des Mathématiques, avec un appendice sur la philosophie des mathématiques de Kant. Par LOUIS COUTURAT. Paris, Felix Alcan, 1905. viii + 310 pp.

THE present work owes its origin, according to the author, to the appearance of "the masterly work of Bertrand Russell which bears the same title." It seems that originally it was intended to be merely a review of the older work. But it is much more than a review of Russell's treatise. We can readily sympathize with the author when he tells us in the preface that in commenting and illustrating Russell's theories he was led gradually to include in his review abstracts of most of the recent papers dealing with the same questions. The result has been that the author has written a comprehensive and careful report on the present state of the logical foundations of mathematics, which on account of its clear style and admirable arrangement of content is valuable, not merely as a work of reference but also as a book well adapted to the needs of anyone desiring to acquaint himself with the fundamental ideas and methods of the subject with which it deals.

The book is divided into six chapters, to which are added two "Notes" and an appendix. The first chapter deals with the principles of formal logic, as developed by Peano and others. The next four chapters discuss respectively the notions of number, order, the continuum, and magnitude. The last and by far the longest chapter deals with the foundations of geometry. It is divided into four parts treating respectively of the dimensions and topology, projective geometry, descriptive geometry, and metric geometry. This should give an idea of the scope of the book. Dealing as it does with an order of ideas that is familiar to the readers of these pages, a more detailed analysis of the contents would be out of place. We regret the absence in the book of an alphabetical subject index.

J. W. YOUNG.

NOTES.

THE Annual Register of the AMERICAN MATHEMATICAL SOCIETY is now in preparation and will be issued in January. Blanks for furnishing necessary information have been sent to the members. Early notice of any changes since the issue of the last Register will greatly facilitate the work of the Secretary.

THE Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will hold two joint sessions with Sections A and D of the American association for the advancement of science on Monday and Tuesday, December 30–31, 1907, at the University of Chicago. The speakers on Monday afternoon will consider the present status of the teaching of mathematics to students of engineering, both in this country and abroad. On Tuesday morning there will be a symposium on the topic: "What is needed in the teaching of mathematics to students of engineering?" The topics included will be (*a*) What branches of mathematics are desirable? (*b*) To what extent should the various branches be taught? (*c*) What methods of presentation are needed? (*d*) What are the chief ends to be attained? Prominent practical engineers and teachers of mathematics and engineering will take part in the programme. There will be a banquet to engineers and mathematicians on Monday evening. Further details will be announced in the printed program, which will be distributed well in advance of the meeting.

TITLES and abstracts of papers intended for presentation before Section A of the American association for the advancement of science at the coming Chicago meeting should be sent to Professor G. A. MILLER, 907 West Nevada street, Urbana, Ill., who has been appointed secretary of Section A to succeed Professor L. G. WELD, who has retired after several years' service in that office.

THE second regular meeting of the Southwestern Section of the AMERICAN MATHEMATICAL SOCIETY will be held at Washington University, St. Louis, November 30, 1907. A meeting of the Central association of teachers of science and mathematics will be held on Friday and Saturday of the same week at the McKinley high school. The programmes have been adjusted for the convenience of those who desire to attend both meetings.

THE concluding (October) number of volume 8 of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "The expansion of a function in terms of normal functions," by MAX MASON; "Sur les opérations linéaires (troisième note)," by MAURICE FRÉCHET; "The elliptic integral in electromagnetic theory," by A. G. GREENHILL; "Notes and errata, volume 8."

Professor MAXIME BÔCHER has been elected editor in chief of the *Transactions*, succeeding Professor E. H. MOORE, who has held this office since the journal was founded in 1899.

THE opening (October) number of volume 9 of the *Annals of Mathematics* contains the following papers: "The calculation of logarithms," by J. K. WHITTEMORE; "Cubic curves in reciprocal triangular situation," by J. F. MESSICK; "The groups generated by two operators such that each is transformed into its inverse by the square of the other," by G. A. MILLER.

AT Parma, September 23–28, the first regular meeting was held and the organization perfected of the Italian association for the advancement of science (Società italiana per il progresso delle scienze). It is composed of fourteen sections, the first including mathematics, astronomy and geodesy. The first president of the association is Professor V. VOLTERRA, of the University of Rome, with Professor M. CERRUTI, of the University of Rome, as president of section A. The following mathematical papers were read at this meeting: By U. AMALDI, "Lie's theory of continuous groups of transformations" (report); by E. BORTOLOTTI, "Plans for the publication of the works of Ruffini"; by P. BURGATTI, "Some points in the theory of differential equations"; by G. FUBINI, "Recent methods for the solution of Dirichlet's problem"; by A. GARBASSO, "Mirage and the optics of heterogeneous media" (report); by G. LAURICELLA, "On functional equations"; by T. LEVI-CIVITA, "On electromagnetic mass" (report); by R. MARCOLONGO, "Mathematical theory of elasticity"; by C. SOMIGLIANA, "On the mathematical preparation of students of engineering"; by O. TEDONE, "Differential equations of mathematical physics"; by G. VAILATI, "Mathematical instruction in the middle schools."

THE educational museum of Teacher's College, Columbia, has prepared a second list of lantern slides illustrating the development of mathematics. Of the present series, Nos. 120–145 are pictures of sample pages of old text-books, etc., Nos. 145–207, old mathematical instruments, Nos. 208–232, modern mechanical calculation, Nos. 253–265, the development of analytics and the calculus. The 160 plates of the present series can be obtained on the same terms as those of the earlier one (see BULLETIN, vol. 13, page 410).

AMONG the books just issued in Professor Cattell's Library of philosophy (Scribner, New York) are translations by Professor G. B. HALSTED of H. Poincaré's two recent monographs: Science and hypothesis (196 pages) and The value of science (147 pages).

THE following books are announced to appear this fall: M. BÔCHER, Higher algebra, Macmillan; E. J. TOWNSEND, Differential and integral calculus, Henry Holt; W. J. BERRY, Differential equations of the first species, Van Nostrand.

UNIVERSITY OF PARIS. *First semester*, beginning November 4. — By Professor G. DARBOUX: Deformation of surfaces, two hours. — By Professor E. GOURSAT: Elements of the theory of analytic functions, two hours. — By Professor P. PAINLEVÉ: General laws of motion and equilibrium, two hours. — By Professor P. APPELL and Dr. E. BLUTEL: General course in mathematics, one hour. — By Professor L. RAFFY: History and methods of analytic geometry, two hours. — By Professor H. POINCARÉ: Theory of the moon, two hours. — By Professor J. BOUSSINESQ: Analytic theory of heat, two hours. — By Professor G. KOENIGS: Thermodynamic theory of machines, two hours. — By Professor E. BOREL: Theory of probabilities, two hours. — By Professor J. TANNERY: Differential and integral calculus, two hours. — By Professor J.

HADAMARD: General mathematics, two hours. Conferences, two hours per week, will be held by Professors Raffy, Hadamard, Puiseux and Drs. Blutel, Servant. *Second semester*, beginning March 10, 1908. — By Professor E. PICARD: Principal types of differential equations of mathematical physics, two hours. — By Professor E. GOURSAT: Partial differential equations, two hours. — By Professor P. PAINLEVÉ: Analytic mechanics, two hours. — By Professor P. APPELL: Analysis and mechanics, two hours. — By Professor L. RAFFY: Differential properties of space curves, two hours. — By Professor P. ANDOYER: Mathematical astronomy, two hours. — By Professor J. BOUSSINESQ: Theory of periodic vibrations, two hours. — By Professor G. KOENIGS: Kinematics, two hours. — By Professor J. TANNERY: Differential and integral calculus, two hours. — By Professor E. BOREL: Mechanics, two hours. — By Professor J. HADAMARD: General mathematics, two hours. The last three courses will be given in the Ecole normal both semesters. Mathematical conferences same as first semester.

PROFESSOR G. MORERA, of the University of Turin, has been elected a member of the royal academy dei Lincei at Rome.

PROFESSOR E. PASCAL, of the University of Pavia, has been appointed professor of higher analysis at the University of Naples.

PROFESSOR G. TORELLI, of the University of Palermo, has been appointed professor of the calculus at the University of Naples.

DR. G. VITALI has been appointed docent in the calculus at the University of Genoa.

AT the recent annual meeting of the Cambridge philosophical society, Dr. E. W. HOBSON was reëlected president and Dr. E. W. BARNES secretary.

PROFESSOR H. M. TORY, of McGill University, has resigned to accept the presidency of the new provincial university of Alberta, Canada.

DR. R. C. MACLAURIN, of the University of Wellington, New Zealand, has been appointed professor of mathematical physics at Columbia University.

DR. J. A. EIESLAND, of the U. S. Naval Academy, has been appointed professor of mathematics at the University of West Virginia.

Dr. S. G. Barton, of the University of Pennsylvania, has been appointed to the chair of mathematics in the Clarkson School of Technology, Potsdam, N. Y.

Dr. J. F. Messick, late of Williams College, has been appointed professor of mathematics at Randolph Macon College, Va.

Dr. W. A. Manning, of Stanford University, has been promoted to an assistant professorship of mathematics.

At the University of Maine, Mr. H. R. Willard has been promoted to an assistant professorship of mathematics.

At Yale University, Mr. E. W. Sheldon, Mr. H. T. Burgess and Mr. J. C. Rayworth have been appointed instructors in mathematics.

At the University of California, Professors G. C. Edwards, C. A. Noble, and A. W. Whitney have been granted leave of absence for one year. Mr. A. J. Champreux has been promoted to an instructorship. Messrs. J. Lipke, J. H. Tener, and B. A. Bernstein have been appointed instructors, and Messrs. E. F. A. Carey and H. W. Stager assistants.

The following academic changes are announced: Mr. E. I. Shepard has returned as instructor in Williams College. Miss C. Busbee has been granted leave of absence from Mount Holyoke College, Mass., for further study. Miss E. N. Martin has returned to Mount Holyoke as instructor in mathematics. Miss A. L. Van Benschoten has returned to her professorship of mathematics at Wells College, Aurora, New York.

Dr. F. H. Loud, professor of mathematics and astronomy at Colorado College since 1877, has retired from active teaching.

Professor Maurice Loewy, director of the Paris observatory and professor of astronomy at the University of Paris, died October 15 at the age of 74 years.

Recent catalogues of second-hand books: A. Lorentz, 10 Kurprinzstrasse, Leipzig, catalogue 165, logic, about 3,000 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Baire (R.). Leçons sur les théories générales de l'analyse. 1$^{\text{er}}$ Tome: Principes fondamentaux. Variables réelles. Paris, Gauthier-Villars, 1907. 8vo. 10 + 232 pp. Fr. 8.00

Ball (W. W. R.). Histoire des mathématiques. Edition française revue et augmentée, traduite sur la 3$^{\text{e}}$ édition anglaise, par L. Freund. Tome 2: Avec des additions de R. de Montessus, suivi de "Les mathématiques modernes depuis Newton jusqu' à nos jours," note complémentaire de G. Darboux. Paris, Hermann, 1907. 8vo. 278 pp.

Carnoy (J.). Cours de géométrie analytique: géométrie de l'espace. 6e édition, entièrement refondue. Paris, 1907. 8vo. 579 pp. Fr. 9.00

Darboux (G.). See Ball (W. W. R.).

Freund (L.). See Ball (W. W. R.).

Kaluza (T.). Die Tschirnhaustransformation algebraischer Gleichungen mit einer Unbekannten. (Diss.) Königsberg, 1907. 8vo. 103 pp.

Lesser (I.). Die Entwickelung des Funktionsbegriffes und die Pflege des funktionalen Denkens im Mathematikunterricht unserer höheren Schulen. Frankfurt, Knauer, 1907. 74 pp. M. 1.80

Loney (S. L.). See Tanner (A. S. G.).

Montessus (R. de). See Ball (W. W. R.).

Osgood (W. F.). A first course in the differential and integral calculus. New York, Macmillan, 1907. 12 mo. 8 + 423 pp. Cloth.

Schulte-Tigges (E.). Ellipse und Parabel als Zentralprojektionen des Kreises. Berlin, Reimer, 1907. Cloth. M. 10.00

——. Die Hyperbel als Zentralprojektion des Kreises. Berlin, Reimer, 1907. Cloth. M. 10.00

Schumacher (H.). Ueber eine Riemannsche Funktionsklasse mit zerfallender Thetafunktion. (Diss.) Strassburg, 1907.

Scott (C. C.). Cartesian plane geometry. Part I: Analytical conics. London, Dent, 1907. 8vo. 442 pp. Cloth. 5s.

Tanner (A. S. G.) and Loney (S. L.). Solutions of the examples in Loney's Coördinate geometry. London, Macmillan, 1907. 8vo. 7s. 6d.

II. ELEMENTARY MATHEMATICS.

Dassen (C. C.). Tratado elemental de algebra. Buenos-Aires, Hermanos, 1907. 8vo. 18 + 528 pp.

Davison (C.) and Richards (C. H.). Plane geometry for secondary schools. Cambridge, University Press, 1907. 8vo. 420 pp. Cloth. 4s.

Ehrig (G.). Arithmetik und Algebra. Ein Lehr- und Uebungsbuch der Gleichungslehre für Baugewerkenschulen und verwandte technische und gewerbliche Lehranstalten mit besonderer Berücksichtigung der praktischen Anwendungen. Mit über 1000 Uebungsaufgaben und zahlreichen ausgeführten Rechenbeispielen. Leipzig, Leineweber, 1907. 8vo. 8 + 164 pp. M. 3.60

Exercices de géométrie comprenant l'exposé des méthodes géométriques et 2000 questions résolues. 4e édition. Paris, Poussielgue, 1907. 8vo. 21 + 1233 pp.

Fourrey (E.). Curiosités géométriques. Paris, 1907. 8 + 431 pp. Fr. 3.50

Gerard (L.). See Niewenglowski (B.).

Gorse (F.). A school algebra course. Part I, to simple simultaneous equations; Part II, factors to quadratic equations; Part III, surds to the binomial theorem. Cambridge, University Press, 1907. 8vo. Part I, 106 pp.; Part II, 102 pp.; Part III, 122 pp. Each 1s.

Guillemin (A.). Tableaux logarithmiques A et B équivalant à des tables de logarithmes à 6 et à 9 décimales, avec notice explicative donnant la théorie et le mode d'emploi des tableaux. Paris, 1907. 8vo. Fr. 4.00

Hupe (A.). See Müller (H.).

Kett (A.). Auflösungen für die Trigonometrie. Für Schule und Praxis herausgegeben. 2te vermehrte und verbesserte Auflage. Neustrelitz, Heydemann, 1907. 8vo. 19 pp. M. 0.60

Mittenzwey (L.). Mathematische Kurzweil oder 333 Aufgaben, Kunststücke, geistanregende Spiele, verfängliche Schlüsse, Scherze, Ueberraschungen und dergleichen aus der Zahlen- und Formenlehre für jung und alt zur Unterhaltung und Belehrung. 5te Auflage. Leipzig, 1907. 108 pp. M. 1.50

Müller (H.). Die Mathematik auf den Gymnasien und Realschulen. Für den Unterricht dargestellt. Ausgabe B: Für reale Anstalten und Reformschulen, unter Mitwirkung von A. Hupe. 2ter Teil. Die Oberstufe. (Lehraufgabe der Klassen Ober-Sekunda und Prima.) Abteilung I: Planimetrie, Algebra, Trigonometrie und Stereometrie. 3te Auflage. Leipzig, Teubner, 1907. 8vo. 10 + 251 pp. M. 2.80

Niewenglowski (B.) et Gerard (L.). Leçons de géométrie élémentaire, conformes aux programmes du 27 juillet 1905, pour les classes de première C et D et de mathématiques A et B. Géométrie plane; géométrie dans l'espace. Paris, Gauthier-Villars, 1907. 8vo. Géométrie plane, 12 + 255 pp.; Géométrie dans l'espace, 334 pp. Fr. 5.00

Peters (J. M.). Die Summen der ganzen positiven Potenzen der natürlichen Zahlen und ihre Darstellung als Funktionen der Summe dieser Zahlen. Köln, 1907. 4to. 24 pp. M. 1.50

Pietzker (F.). Lehrgang der Elementar-Mathematik. Teil I: Unterstufe. Leipzig, Teubner, 1907. 8vo. 318 pp. M. 3.20

Reidt (F.). Anleitung zum mathematischen Unterricht an höheren Schulen. 2te Auflage, revidiert und mit Anmerkungen versehen von H. Schotten. Berlin, Grote, 1907. 8vo. 269 pp.

Richards (C. H.). See Davison (C.).

Schotten (H.). See Reidt (F.).

Schubert (H.) und Schumpelick (A.). Arithmetik für Gymnasien. 1$^{\text{es}}$ Heft. Für mittlere Klässen. Leipzig, Göschen, 1907. M. 1. 80

——. Ausgewählte Resultate zur Arithmetik für Gymnasien. Leipzig, Göschen, 1907. M. 0. 60

Schumpelick (A.). See Schubert (H.).

Tanner (J. H.). High school algebra. New York, American Book Company, 1907. 8vo. 6 + 346 pp. Cloth. $1.00

Tarry (G.). Tablettes des cotes-relatives à la base 20580 des facteurs premiers d'un nombre inférieur à N et non divisible par 2, 3, 5, ou 7. Paris, Gauthier-Villars, 1907. 8vo. Fr. 1.25

Young (E.). A first year's course in geometry and physics. London, Bell, 1907. 8vo. 182 pp. 2s. 6d

III. APPLIED MATHEMATICS.

Appell (P.) et Chappuis (J.). Leçons de mécanique élémentaire, à l'usage des élèves des classes de mathématiques A et B, conformément aux programmes de 1905. 2$^{\text{e}}$ édition, entièrement refondue. 1$^{\text{re}}$ partie : Notions géométriques. Cinématique ; 2$^{\text{e}}$ partie : Dynamique et statique du point. Statique des corps solides. Machines simples. Paris, Gauthier-Villars, 1907. 2 volumes. 16mo. 1$^{\text{re}}$ partie, 9 + 192 pp. Fr. 2.75. 2$^{\text{e}}$ partie, 244 pp. Fr. 3.25

Barkhausen (H.). Das Problem der Schwingungserregung mit besonderer Berücksichtigung schneller elektrischer Schwingungen. Leipzig, 1907. 4 + 113 pp. M. 4.00

Chappuis (J.). See Appell (P.).

Desportes (E.). Eléments de géométrie descriptive. Nouvelle édition entièrement refondue, conforme aux programmes du 27 juillet 1905; classes de première C et D, et de mathématiques A et B. Paris, Colin, 1907. 8vo. 332 pp. Fr. 4.00

Eggert (O.). Einführung in die Geodäsie. Leipzig, 1907. 10 + 437 pp. M. 10.00

Friedel (J.). See Lamb (H.).

Hauber (W.). Statik. Neudrucke. Leipzig, 1907. 1$^{\text{er}}$ Teil, 148 pp. 2$^{\text{er}}$ Teil, 148 pp. Each, M. 0.80

Konen (H.). See Schuster (A.).

Kuenen (J. P.). Die Zustandsgleichung der Gase und Flüssigkeiten und die Kontinuitätstheorie. Braunschweig, 1907. 10 +241 pp. M. 6.50

Lamb (H.). Lehrbuch der Hydrodynamik. Deutsche autorisierte Ausgabe (nach der 3$^{\text{en}}$ englischen Auflage) besorgt von J. Friedel. (B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. XXVI Band.) Leipzig, Teubner, 1907. 8vo. 14 + 788 pp. Cloth. M. 20.00

Laville (C.). See Valbreuze (R. de).

MARTIN (L. A.). Textbook of mechanics. Vol. II. Kinematics and kinetics. New York, Wiley, 1907. 12mo. 4 + 214 pp. Cloth. $1.50

MÜLLER-BRESLAU (H.). Die graphische Statik der Baukonstruktionen. 2ter Band. 1te Abteilung. 4te, vermehrte Auflage. Stuttgart, 1907. 8 + 484 pp. M. 16.00

PERRY (J.) Applied mechanics. A treatise for the use of students, etc. New edition, revised and enlarged. London, Cassell, 1907. 8vo. 696 pp. Cloth. 7s. 6d.

SCHUSTER (A.). Einführung in die theoretische Optik. Autorisierte deutsche Ausgabe, übersetzt von H. Konen. Leipzig, 1907. 8vo. 14 + 413 pp. Cloth. M. 12.00

VALBREUZE (R. DE) et LAVILLE (C.). Eléments de mécanique et de l'électricité. Paris, Dunod, 1907. 16mo. 6 + 379 pp. Fr. 7.00

WALL (C.). Eléments de mécanique appliquée et d'hydrostatique. Notions sommaires sur la résistance des matériaux et sur la graphique, statique à l'usage des élèves, officiers de la marine et des élèves des écoles de maistrance des arsenaux. Paris, Challamel, 1907. 8vo. 269 pp.

LIBRARY

THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and thirty-fifth regular meeting of the Society was held in New York City on Saturday, October 26, 1907, a single morning session sufficing for the unusually brief programme. The attendance included the following twenty-eight members of the Society:

Professor G. A. Bliss, Professor Maxime Bôcher, Professor Joseph Bowden, Professor E. W. Brown, Dr. J. E. Clarke, Professor F. N. Cole, Miss L. D. Cummings, Mr. G. W. Hartwell, Professor E. V. Huntington, Mr. S. A. Joffe, Mr. E. H. Koch, Dr. G. H. Ling, Professor E. O. Lovett, Mr. E. B. Lytle, Professor Max Mason, Professor Mansfield Merriman, Dr. R. L. Moore, Mr. H. W. Reddick, Professor L. W. Reid, Mr. F. H. Smith, Professor P. F. Smith, Professor H. D. Thompson, Mr. C. A. Toussaint, Professor E. B. Van Vleck, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Professor J. W. Young.

Vice-President Professor P. F. Smith occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. V. R. Aiyar, Gooty, India; Professor P. P. Boyd, Hanover College, Ind.; Dr. Charles Haseman, Indiana University; Professor C. A. Proctor, Dartmouth College; Mr. J. M. Rysgaard, University of North Dakota; Mr. C. A. Toussaint, College of the City of New York. Thirteen applications for admission to membership in the Society were received.

A list of nominations of officers and of the members of the Council was adopted and ordered placed on the official ballot for the annual election at the December meeting.

The following papers were read at this meeting:

(1) Professor R. D. CARMICHAEL: "A certain class of quartic curves."

(2) Professor R. D. CARMICHAEL: "Geometric properties of quartic curves possessing fourfold symmetry with respect to a point."

(3) Professor OSWALD VEBLEN: "On magic squares."

(4) Professor L. E. Dickson: "On triple algebras and ternary cubic forms."

(5) Sir G. H. Darwin: "Further note on Maclaurin's spheroid."

(6) Dr. J. L. Coolidge: "The equilong transformations of space."

(7) Professor Edward Kasner: "Note on isothermal systems."

(8) Dr. R. L. Moore: "A note concerning Veblen's axioms for geometry."

(9) Professor Joseph Bowden: "Proof of a formula in combinations."

Professor Darwin's paper was communicated to the Society through Professor E. B. Van Vleck. In the absence of the authors, the papers of Professor Carmichael, Professor Dickson, Professor Darwin, and Dr. Coolidge were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper Professor Carmichael studies the general nature of a class of quartic curves whose equation may be represented thus: Set

$$m_i = c_i \sqrt{[(x - a_i)^2 + (y - b_i)^2]} \quad (i = 1, 2, 3),$$

in which $c_i \neq 0$ and the radical is to be taken with the positive sign. Then the equation of the curve studied takes the form

$$(m_1 + m_2 + m_3)(m_1 + m_2 - m_3)(m_1 - m_2 + m_3)(m_1 - m_2 - m_3) = 0,$$

which clears of radicals when the parentheses are removed. A method is also given for the construction of the curve by continuous motion when the c's are commensurable.

2. In a previous communication Professor Carmichael has shown that plane algebraic curves possessing fourfold symmetry with respect to a point divide into two groups. In the present paper he studies the geometric properties of each of these two classes for the case of quartic curves. Special attention is given to the forms of the loci, and the number of singularities in each case is completely determined.

3. Professor Veblen's note shows how it is possible to derive magic squares with p^n elements in a line from the corresponding finite euclidean geometries. The method is a generalization of that of G. Arnoux: Arithmétique graphique. Les espaces arithmétiques hypermagiques, Paris, 1894.

4. Professor Dickson's paper appears in full in the present number of the BULLETIN.

5. In volume 4 of the *Transactions*, Sir George Darwin showed that the form of Maclaurin's ellipsoid could be determined by spherical harmonics to a higher order of approximation than had usually been supposed possible. In the paper referred to he had been unable to determine the angular velocity corresponding to a given ellipticity as far as the cube of that ellipticity. In the present paper he shows that if certain terms previously omitted were retained the defect can be remedied.

6. Scheffers has given the name "equilong" (German, aequilong) to those transformations of the plane which carry a line into a line, keeping invariant the distance between the points of contact of a line with any two envelopes which it touches. These transformations are, in a sense, dual to the conformal ones, and Scheffers has shown that, whereas the latter depend on an arbitrary function of the usual complex variable, the former may be expressed by an arbitrary function of a complex variable of different type. Dr. Coolidge's paper dealt with the corresponding transformations in three dimensions, namely, those which carry a plane into a plane, keeping invariant the distance of the points of contact with any two envelopes which the plane may touch. In this case the duality between conformal and equilong transformations exists in the statement only, for whereas the former group depends on ten arbitrary parameters, the latter involves two arbitrary functions.

7. Professor Kasner's paper appears in full in the present number of the BULLETIN.

8. In volume 5, number 3, pages 343–384 of the *Transactions*, Professor Veblen has exhibited a set of axioms for geometry in terms of point and order. Dr. Moore shows that Axiom II becomes redundant if Axiom V is strengthened by assuming that C is different from B.

9. Professor Bowden gave an elementary proof by mathematical induction of the formula

$$C_r^{m+n} = \sum_{k=1}^{k=r+1} C_{r-k+1}^{m} C_{k-1}^{n}.$$

F. N. COLE,
Secretary.

ON TRIPLE ALGEBRAS AND TERNARY CUBIC FORMS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, October 26, 1907.)

1. FOR any field F in which there is an irreducible cubic equation $f(\rho) = 0$, the norm of $x + y\rho + z\rho^2$ is a ternary cubic form C which vanishes for no set of values x, y, z in F, other than $x = y = z = 0$. The conditions under which the general ternary form has the last property are here determined for the case of finite fields. One formulation of the result is as follows:

THEOREM. *The necessary and sufficient conditions that a ternary cubic form C shall vanish for no set of values x, y, z in the $GF[p^n]$, $p > 2$, other than $x = y = z = 0$, are that its Hessian shall equal mC, where m is a constant different from zero, and that the binary form obtained from C by setting $z = 0$ shall be irreducible in the field.*

Although I have not hitherto published a proof of this theorem, I have applied it to effect a determination* of all finite triple linear algebras in which multiplication is commutative and distributive, but not necessarily associative, while division is always uniquely possible. I shall here (§ 11) determine these algebras by applying directly the more fundamental conditions from which the preceding theorem is derived.

These ternary cubic forms arise in various other problems; for instance, in the normalization of families of ternary quadratic forms containing three linearly independent forms.

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* *Amer. Math. Monthly*, vol. 13 (1906), pp. 201–205. References are there given to my earlier papers on the subject.

All such ternary cubic forms in a finite field are equivalent under linear transformation in the field (§ 9).

2. Let a ternary form,* with coefficients in the $GF[p^n]$,

$$(1)\qquad C \equiv ax^3 + bx^2y + cx^2z + dxy^2 + exz^2 + fxyz + gy^3 + hy^2z + kyz^2 + lz^3,$$

vanish in the field only for $x = y = z = 0$. Then for any assigned values, not both zero, of y and z in the field, the cubic in x is irreducible; hence it has three roots in the $GF[p^{3n}]$, whose product is $K \equiv -a^{-1}(gy^3 + \cdots + lz^3)$. Now K is irreducible in the $GF[p^n]$. In the $GF[p^{3n}]$ every set of three factors of K, conjugate with respect to the $GF[p^n]$, may be given the form $\gamma(y - \rho z)$, $\gamma^{p^n}(y - \rho^{p^n}z)$, $\gamma^{p^{2n}}(y - \rho^{p^{2n}}z)$, where γ is a root of

$$\gamma^{p^{2n}+p^n+1} = -a^{-1}g.$$

It follows that (1) vanishes for $x = \gamma(y-\rho z)$. In other words, C must have a linear factor, in the $GF[p^{3n}]$,

$$(2)\qquad x - \lambda y - \mu z.$$

For $x = \lambda y + \mu z$, let C become

$$R_0y^3 + R_1y^2z + R_2yz^2 + R_3z^3.$$

Then for y and z arbitrary in the $GF[p^n]$, this sum must vanish for suitably chosen values of λ and μ in the $GF[p^{3n}]$. Hence the four equations $R_i = 0$ must be solvable simultaneously in the $GF[p^{3n}]$.

The conditions $R_i = 0$ are seen to be

$$(3)\ R_0 \equiv a\lambda^3 + b\lambda^2 + d\lambda + g = 0,\quad (4)\ R_3 \equiv a\mu^3 + c\mu^2 + e\mu + l = 0,$$

$$(5)\ R_1 \equiv R_0'\mu + c\lambda^2 + f\lambda + h = 0,\quad (6)\ R_2 \equiv R_3'\lambda + b\mu^2 + f\mu + k = 0,$$

where the accents denote differentiation. If (3) had a root in the $GF[p^n]$, C would vanish for $x = \lambda$, $y = 1$, $z = 0$. Hence (3) and (4) must be irreducible in the $GF[p^n]$. Thus $R_0' \neq 0$, $R_3' \neq 0$.

For λ a root of (3) and for μ defined by (5), we seek the condition under which (2) vanishes for a set of elements x, y, z, not all zero, in the $GF[p^n]$. Eliminating μ between

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* To include the cases $p = 2$, $p = 3$, we do not prefix binomial coefficients.

$$x - \lambda y - \mu z = 0$$

and (5), and then eliminating λ^3 by (3), we obtain

$$(7)\quad \lambda^2(3ax + by + cz) + \lambda(2bx + 2dy + fz) + dx + 3gy + hz = 0,$$

Hence such elements x, y, z do not exist if, and only if,

$$(8)\qquad \begin{vmatrix} 3a & b & c \\ 2b & 2d & f \\ d & 3g & h \end{vmatrix} \neq 0.$$

THEOREM. *The necessary and sufficient conditions that* C *shall vanish for no set of values* x, y, z *in the* $GF[p^n]$, *other than* $x = y = z = 0$, *are that* C *shall have a linear factor* (2) *in the* $GF[p^{3n}]$, *that* (3) *shall be irreducible in the* $GF[p^n]$, *and that* (8) *shall hold.*

3. We readily deduce the theorem of § 1 from the preceding theorem. When the conditions of the latter theorem are satisfied, C has three distinct linear factors in the $GF[p^{3n}]$ and hence can be transformed into $\xi\eta\zeta$. The Hessian of the latter is $2\xi\eta\zeta$. In view of the covariance of the Hessian, we conclude that the Hessian of C is of the form mC, m an element $\neq 0$ of the $GF[p^n]$, $p > 2$. Conversely when the Hessian has this property, C has three distinct linear factors.* Each factor is of the form (2), where λ is a root of the irreducible equation (3) and μ is uniquely determined by (5), so that λ and μ belong to the $GF[p^{3n}]$. Further, (8) is satisfied when $m \neq 0$. Indeed, the coefficients of (7) equal $\frac{1}{2}C_{xx}$, C_{xy}, $\frac{1}{2}C_{yy}$, respectively; if they all vanished, the Hessian would vanish.

4. Although the conditions on the coefficients of C may be obtained from the Hessian, we deduce them in convenient form directly from the conditions for the simultaneity of the four equations $R_i = 0$ in the $GF[p^{3n}]$.

For any field we may make $b = 0$ in (1) by an obvious transformation on y and z. Moreover the case $b = 0$ is sufficient for the applications to linear algebras.

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*For a direct proof for finite fields, see §§ 5, 6. In the algebraic theory of ternary cubic forms, this property follows from the canonical types (cf. Gordan, *Transactions*, vol. 1, p. 403). We note that if the Hessian of $x^3 + y^3 + z^3 + 6mxyz$ is a multiple of the form, then $m = -\frac{1}{2}\omega$, where $\omega^3 = 1$. Replacing ωz by Z, we obtain the factors $x + y + Z$, $x + \omega y + \omega^2 Z$, $x + \omega^2 y + \omega Z$.

Eliminating μ between (5) and (6) and then eliminating the higher powers of λ by means of (3), we obtain a quadratic function of λ, which must vanish identically, in view of the irreducibility of (3). Hence

$$(9) \qquad 3aJ = 0, \quad dJ = 0, \quad 3aK + dL = 0,$$

where

$$(10) \qquad \begin{aligned} J &= afh - adk - 3aeg + c^2g, \\ K &= ah^2 - 3agk + cfg - cdh, \\ L &= ach + 4ade - af^2 - c^2d. \end{aligned}$$

But for $b = 0$, (8) becomes

$$(11) \qquad 6adh - 9afg - 2cd^2 \neq 0.$$

Thus $3a$ and d do not both vanish. Hence

$$(12) \qquad J = 0.$$

Eliminating λ between (5) and (6), and dividing by R_3, we obtain the quotient

$$(13) \quad Q \equiv 9a^2d\mu^2 + (3acd + 9a^2h)\mu + c^2d + 3ach - 3ade,$$

and a remainder of degree two, which must vanish identically, in view of the irreducibility of $R_3 = 0$. Hence

$$(14) \quad cL - 3aM = 0, \quad eL + 3aN = 0, \quad eM + cN = 0,$$

where

$$(15) \quad M = 3adl - afk + aeh, \quad N = ak^2 - 3ahl - cdl.$$

We proceed to prove that, if Q is not identically zero, and if conditions (9), (11), (14) are satisfied, and if (3) is irreducible in the $GF[p^n]$, then the four equations $R_i = 0$ are simultaneous in the $GF[p^{3n}]$. Let λ be a root of (3) and μ be defined by (5). In view of the origin of (9), μ satisfies (6). In view of the origin of (14), μ satisfies $QR_3 = 0$. It remains only to show that $Q \neq 0$. First, μ is not an element of the $GF[p^n]$. For, if so, equation (5) and the irreducibility of (3) would give

$$3a\mu + c = 0, \quad f = 0, \quad d\mu + h = 0,$$

and determinant (8) would vanish, having proportional elements in the first and third columns. Next, a mark μ of the $GF[p^{3n}]$, not in the $GF[p^n]$, cannot belong to the $GF[p^{2n}]$. Hence $Q \neq 0$.

5. Finally, let Q be identically zero. The case $p = 3$ is excluded, since then $cd \neq 0$ by (11). Hence $d = h = 0$. Then $fg \neq 0$ by (11). By (9),

$$c^2 = 3ae, \quad cf = 3ak. \tag{16}$$

Then (14) are satisfied, viz., the result of eliminating λ between (5) and (6) now vanishes identically. Removing the factor λ in (5), we now have

$$\mu = (-c\lambda - f)/3a\lambda. \tag{17}$$

Substituting this value in (4), eliminating λ^3 by means of (3), and e by means of (16), we get

$$27a^2gl = c^3g - af^3. \tag{18}$$

In the resulting form C satisfying (16), (18) and

$$b = d = h = 0, \quad fg \neq 0 \qquad (p \neq 3), \tag{19}$$

we replace x by $X - \frac{1}{3}ca^{-1}z$ and obtain

$$aX^3 + gy^3 - \tfrac{1}{27}a^{-1}g^{-1}f^3z^3 + fXyz. \tag{20}$$

Its Hessian is seen to equal $-6f^2C$. Let

$$g = -a\nu, \quad y = -Y, \quad fz = 3a\nu Z.$$

Then by (20) and (3),

$$C = a(X^3 + \nu Y^3 + \nu^2 Z^3 - 3\nu XYZ), \tag{21}$$

where $\lambda^3 = \nu$ is irreducible in the $GF[p^n]$, whence $p^n = 3m + 1$. After the present change of notation is made, (2) becomes, in view of (17),

$$X + \lambda Y + \lambda^2 Z. \tag{22}$$

This is indeed a factor of (21) for $\lambda^3 = \nu$. Connected with this form (21) is a remarkable non-linear algebra in three units in which division is always uniquely possible.*

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* Dickson, *Göttinger Nachrichten*, 1905, p. 359, p. 373.

6. Returning to the case in which Q is not identically zero, we treat first the case in which the modulus exceeds 3. Then we may transform* (1) into a form having $b = c = f = 0$. Condititions (9)–(15) then reduce to $dh \neq 0$ and

$$(23)\qquad \begin{aligned} dk + 3eg &= 0, & 4de^2 + 3ak^2 - 9ahl &= 0, \\ eh + 3dl &= 0, & 4d^2e + 3ah^2 - 9agk &= 0. \end{aligned}$$

The second may be derived from the other three. In view of these conditions the Hessian of C reduces to $3deC$. Evidently $e \neq 0$.

Conversely, if the Hessian $3adex^3 + \cdots$ of a form C, having $b = c = f = 0$, is a non-vanishing multiple of C, then conditions (23) follow. Also $h \neq 0$. For, if $h = 0$, conditions (23) give

$$l = 0, \quad d = \frac{-3ak^2}{4e^2}, \quad g = \frac{ak^3}{4e^3},$$

and (3) would vanish for $\lambda = -k/e$.

7. We may readily enumerate the resulting forms C. We consider first the case $p > 3$, and set $\epsilon = 1$ if $p^n = 3m + 1$, $\epsilon = 0$ if $p^n = 3m + 2$. We set $b = c = 0$, thus considering one of p^{2n} coordinate cases. Let first $Q \equiv 0$, so that d, h, e, k all vanish (§ 5). There are $\frac{2}{3}\epsilon(p^n - 1)^2$ sets a, g for which $a\lambda^3 + g = 0$ is irreducible in the $GF[p^n]$. Now f may have any value $\neq 0$; while l is determined by (18). Hence there are

$$\tfrac{2}{3}\epsilon(p^n - 1)^3 p^{2n} \text{ forms with } Q \equiv 0.$$

For $Q \not\equiv 0$, d and h are not both zero. Let first $d = 0$. For each of the $\frac{2}{3}\epsilon(p^n - 1)^2$ sets a, g, the coefficients h and e may have any values not zero, f being not zero by (11). Then $f = 3egh^{-1}$, $k = \frac{1}{3}g^{-1}h^2$ by $J = K = 0$, and M is then zero. Finally, $eL + 3aN = 0$ determines l. Next, for $d \neq 0$, we make $f = 0$ and apply § 6. The number of irreducible cubics $a\lambda^3 + d\lambda + g$ with $d \neq 0$ is†

$$\kappa \equiv \tfrac{1}{3}(p^{2n} - 1)(p^n - 1) - \tfrac{2}{3}\epsilon(p^n - 1)^2.$$

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[*] First by $x' = x + \rho y + \sigma z$ we make $b = c = 0$. The coefficient of x is a binary quadratic form, so that the term fyz may be deleted.

[†] BULLETIN, October, 1906, p. 4.

For each set a, d, g, and for any $h \neq 0$, (23) give

$$l = \frac{-eh}{3d}, \quad k = \frac{-3eg}{d}, \quad e(4d^3 + 27ag^2) = -3adh^2,$$

the coefficient of e being the discriminant of the irreducible cubic. In view of b, c, f, we have the factor p^{3n}. Hence there are

$$\tfrac{2}{3}\epsilon(p^n - 1)^4 p^{2n} + \kappa(p^n - 1)p^{2n} \text{ forms with } Q \neq 0.$$

THEOREM.* *The total number of ternary cubic forms in the* $GF[p^n]$ *which vanish in the field only for* $x = y = z = 0$ *is*

$$\tfrac{1}{3}(p^{2n} - 1)(p^n - 1)^2 p^{3n}. \tag{24}$$

8. Consider the automorphs of one of our ternary forms C. In view of (3) and (5), we find that $\mu = r\lambda^2 + s\lambda + t$, where $r \neq 0$. Now $C = L_1 L_2 L_3$, where

$$L_1 = x - \lambda y - \mu z, \quad L_2 = x - \lambda^{p^n} y - \mu^{p^n} z, \quad L_3 = x - \lambda^{p^{2n}} y - \mu^{p^{2n}} z. \tag{25}$$

The determinant D of the coefficients in the L's is a mark $\neq 0$ of the $GF[p^n]$, since $D^{p^n} = D$. Let $L'_1 = x' - \lambda y' - \mu z'$, etc. The transformation

$$L'_1 = \tau L_1, \quad L'_2 = \tau^{p^n} L_2, \quad L'_3 = \tau^{p^{2n}} L_3, \tag{26}$$

yields x', y', z' as functions of x, y, z with coefficients in the $GF[p^n]$. Hence there are $p^{2n} + p^n + 1$ automorphs (26) of C. Further,

$$L'_1 = L_2, \quad L'_2 = L_3, \quad L'_3 = L_1 \tag{27}$$

and its square are automorphs of C. Evidently every automorph is generated by (26) and (27).

THEOREM. *The number of automorphs of* C *is* $3(p^{2n} + p^n + 1)$.

9. In view of the order of the general ternary linear homogeneous group in the $GF[p^n]$ and the preceding theorem, it follows that a form C is one of

$$\frac{(p^{3n} - 1)(p^{3n} - p^n)(p^{3n} - p^{2n})}{3(p^{2n} + p^n + 1)} = \tfrac{1}{3}(p^{2n} - 1)(p^n - 1)^2 p^{3n}$$

conjugates. But this number is the same as (24).

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* Another proof results from an enumeration of the distinct products of three linear forms in the $GF[p^{3n}]$ conjugate with respect to the $GF[p^n]$.

THEOREM. *In the* $GF[p^n]$, *all ternary cubic forms which vanish only for* $x = y = z = 0$ *are equivalent under linear transformation.*

10. We consider briefly the case* $p = 2$, the Hessian of C being then identically zero. By an obvious transformation we may make $b = c = h = 0$. Then $afg \neq 0$ by (11). Since it remains only to treat the case in which (18) does not vanish identically, we may set $d \neq 0$. Conditions (9)–(14) then reduce to

$$dk = eg, \quad agk = df^2, \quad dl = fk, \quad ef^2 = ak^2,$$

the last being superfluous. We may determine ρ so that $d\rho^2 \doteq a$. We set

$$x = X, \quad y = \rho Y, \quad z = \frac{ga}{fd} Z, \quad \gamma = \frac{g\rho}{d}.$$

Then C has the factor a, which may be made unity by applying a transformation (26). The complementary factor is

$$(28) \quad X^3 + XY^2 + XZ^2 + \gamma XYZ + \gamma Y^3 + \gamma YZ^2 + \gamma^2 Z^3.$$

Multiplying (5) by λ and applying (3), we get

$$g\mu = f\lambda^2.$$

Let $\tau = \rho\lambda$. Then (3) becomes

$$(29) \qquad \tau^3 + \tau + \gamma = 0.$$

The factor (2) of C is seen to equal

$$(30) \qquad X + \tau Y + \tau^2 Z.$$

By a preliminary transformation on x and y, the irreducible cubic (29) may be transformed into any particular one.

THEOREM. *In the* $GF[2^n]$, *every ternary cubic form which vanishes only for* $x = y = z = 0$ *may be transformed into* (28), *where* γ *is a particular mark for which* (29) *is irreducible.*

11. We proceed to determine all finite triple linear algebras in which multiplication is commutative and distributive, but

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* For $p = 3$, I have determined canonical types of all ternary cubic forms. The results are to appear shortly in the *American Journal.*

not necessarily associative, while division is always uniquely possible. We may assume (*Göttinger Nachrichten*, l. c.) that the units are 1, i, j, where

$$(31)\qquad i^2=j,\quad ij=ji=g-di,\quad j^2=h+\delta i+Dj,$$

x^3+dx-g being irreducible in the $GF[p^n]$. In

$$(x+yi+zj)(\xi+\eta i+\zeta j)=P+Qi+Rj,$$

the determinant of the coefficients of ξ, η, ζ in P, Q, R is of the form (1) with

$$(32)\qquad \begin{gathered} a=1,\quad b=0,\quad c=D-d,\quad e=-h-dD,\\ f=-\delta-2g,\quad k=-gD,\quad l=\delta g+dh, \end{gathered}$$

the coefficients g, d, h being the same in the two forms. Let λ be a root, in the $GF[p^{3n}]$, of (3), viz.,

$$(3')\qquad \lambda^3+d\lambda+g=0.$$

Thus $-\lambda$ plays a rôle analogous to the unit i of the algebra. We may regard λ, d, g, h, δ, D to be of dimensions 1, 2, 3, 4, 3, 2 respectively. Hence we shall set*

$$(33)\qquad h=\epsilon d^2,\quad \delta=\tau g,\quad D=\kappa d,$$

ϵ, τ, κ being of dimension zero. Then, by (5),

$$-\mu=[(\kappa-1)d\lambda^2-(\tau+2)g\lambda+\epsilon d^2]\div(3\lambda^2+d).$$

The numerator is of dimension 4, the denominator 2. Hence

$$(5')\qquad -\mu=\rho\lambda^2+\sigma d,$$

where ρ, σ are of dimension zero. Equating the two values and reducing by (3′), we obtain

$$(34)\qquad \kappa-1=3\sigma-2\rho,\quad \tau+2=3\rho,\quad \epsilon=\sigma.$$

Next, (6) becomes

$$[3\mu^2+2(\kappa-1)d\mu-(\kappa+\epsilon)d^2]\lambda-(\tau+2)g\mu-\kappa dg=0.$$

Eliminating μ by (5′), reducing by (3′), and applying (34), we get

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* The case $d=0$ may be avoided by a transformation of units.

$$(35)\qquad \begin{aligned} 4\rho\sigma - \rho^2 - 3\sigma^2 - 4\sigma + 2\rho - 1 &= 0, \\ (\rho - 1)(\rho - 1 - 3\sigma) &= 0, \end{aligned}$$

the coefficient of λ^2 being zero. For $\rho = 1$, $\sigma = 0$, and the algebra is a field. For $\rho = 1 + 3\sigma$, (35_1) is satisfied; then $\kappa = -\rho$. Substituting (5′) in (4) and reducing by (3′), we find that the coefficients of λ^2 and λ vanish, and that the constant term is

$$-\sigma^2(\sigma + 1)(4d^3 + 27g^2) = 0.$$

But the second factor is not zero in view of the irreducibility of (3′). For $\sigma = 0$, the algebra is a field. For $\sigma = -1$, $\rho = -2$, and we obtain the non-field algebra

$$(36)\qquad i^2 = j, \quad ij = ji = g - di, \quad j^2 = -d^2 - 8gi + 2dj.$$

THE UNIVERSITY OF CHICAGO,
September, 1907.

ISOTHERMAL SYSTEMS IN DYNAMICS.

BY PROFESSOR EDWARD KASNER.

(Read before the American Mathematical Society, October 26, 1907.)

CONSIDER any simply infinite system of plane curves defined by its differential equation

$$(1)\qquad y' = f(x, y).$$

The ∞^2 isogonal trajectories satisfy the equation *

$$(2)\qquad y'' = (F_x + y'F_y)(1 + y'^2),$$

where

$$F = \tan^{-1} f.$$

The theorem of Cesàro-Scheffers states that the trajectories passing through a given point have circles of curvature forming a pencil. We inquire whether any hyperosculating circles exist.

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* Primes are employed to denote derivatives with respect to x, and literal subscripts to denote partial derivatives.

The discussion is valid for the general type

$$(3)\qquad y'' = (\psi - y'\phi)(1 + y'^2)$$

which arises in dynamics* and includes (2) as a special case. Denoting the radius of curvature by r and the arc by s, we find, for the curves (3)

$$(4)\qquad r = \frac{\sqrt{1 + y'^2}}{\psi - y'\phi},$$

$$(5)\qquad \frac{dr}{ds} = \frac{(\phi_y - \phi\psi)y'^2 + (\phi_x - \psi_y + \psi^2 - \phi^2)y' - \psi_x + \phi\psi}{(\psi - y'\phi)^2}.$$

The condition for four-point contact is the vanishing of (5). Hence for a doubly infinite system of type (3) there are at each point two curves which have contact of the third order with their circles of curvature (at that point). The directions of these curves are given by

$$(6)\quad (\phi_y - \phi\psi)y'^2 + (\phi_x - \psi_y + \psi^2 - \phi^2)y' - \psi_x + \phi\psi = 0.$$

The condition that these directions be orthogonal is

$$(7)\qquad \phi_y - \psi_x = 0.$$

Applying this to the isogonal type (2), we find

$$(8)\qquad F_{xx} + F_{yy} = 0,$$

which is recognized as the condition that (1) be isothermal. We have thus obtained a general property of isogonals and a concrete test for an isothermal system.

The isogonal trajectories of any given simply infinite system of curves are such that through each point there pass two with hyperosculating circles at that point.

These circles intersect orthogonally, for every point of the plane, when and only when the given system is isothermal.

The existence of such orthogonal circles is one of the essential properties of dynamical trajectories produced by conservative forces. When condition (7) is fulfilled, the force whose components are ϕ, ψ is conservative, and the totality of ∞^3 possible trajectories may be decomposed into ∞^1 "natural families," one for each value of the energy constant. Such a family is defined then by an equation of type (3) with condition (7) fulfilled.

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* In the study of certain loci termed velocity curves by the author.

The isogonals of a simple system of curves form a "natural family" when and only when the given system is isothermal.

The explicit relation may be obtained as follows: If the work function defining the field is denoted by W, and the constant of energy is taken to be zero, the natural family is represented by

$$(9) \qquad y'' = \tfrac{1}{2}(W_y - y' W_x)(1 + y'^2)/W.$$

If this is to be identical with the isogonal system (2), we must have

$$(10) \qquad F_x = \tfrac{1}{2} W_y/W, \qquad F_y = -\tfrac{1}{2} W_x/W.$$

These are seen to be the conditions that the functions F and $\log \sqrt{W}$ be conjugate harmonic. It follows that the base system of the isogonals is isothermal, and that the corresponding field of force is defined by a work function of the form

$$(11) \qquad W = e^{2H},$$

where H is the conjugate harmonic of F, that is, $F + iH$ is a function of $x + iy$. Our result may now be recast as follows:

A natural family of dynamical trajectories can be identified with a system of isogonal trajectories when and only when the work function is the exponential of an harmonic function.

The field of force associated with a given isothermal system is determined up to a constant factor. The lines of force are the orthogonals of the equipotential lines $W =$ const. and are therefore represented by $F =$ const.; they are seen to be the slope lines of the given system, *i. e.*, the lines joining the points where the curves (1) have parallel tangents. If only the lines of force (themselves necessarily isothermal) are given, the function F may be replaced by any function of the form $c_1F + c_2$, and the work function is determined except for a constant factor and a constant exponent.

The most immediate application of our results is to central forces varying according to the nth power of the radius vector. This is in fact the only case in which a central force can be of type (11). Using polar coordinates we have

$$(12) \qquad W = \rho^{n+1}, \quad H = \tfrac{1}{2}(n+1)\log\rho, \quad F = -\tfrac{1}{2}(n+1)\theta.$$

Among the orbits produced by a central force varying as the nth power of the radius vector are included the isogonal trajectories of the curves

$$y' = -\tan \tfrac{1}{2}(n+1)\theta. \tag{13}$$

This construction yields in the case $n = -2$ (Newtonian law) parabolas with focus at the origin; in the case $n = 1$ (elastic law) equilateral hyperbolas with center at the origin; in the case $n = -5$ the circles through the origin; and in the case $n = -3$ equiangular spirals with pole at the origin.

COLUMBIA UNIVERSITY.

ON THE EQUATIONS OF QUARTIC SURFACES IN TERMS OF QUADRATIC FORMS.

BY DR. C. H. SISAM.

(Read before the American Mathematical Society, September 5, 1907.)

THE quartic surfaces whose equations are of the form

$$\phi_2(A, B, C, D) = 0,$$

where ϕ_2, A, B, C and D are quaternary quadratic forms, were the subject of a paper by H. Durrande in the *Nouvelles Annales*.* By counting the number of constants involved, Durrande concluded that the most general quartic surface could be represented by an equation of the above form. He recognized, however, that his reasoning was not rigorous.

It will here be shown that the coefficients of the quartic surface determined by this equation are not independent, but are subject to a single condition. It will also be shown that *the equation of a general quartic surface can be written in the form*

$$\theta_2(A, B, C, D, E) = 0,$$

where θ_2 is a quinary quadratic form.

Let $\phi_2 = 0$ be reduced to the form

$$A^2 + B^2 + C^2 + D^2 = 0, \tag{1}$$

where

$$A \equiv \Sigma a_{ij} x_i x_j \qquad (i \leqq j \leqq 4)$$

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* Durrande, *Nouvelles Annales*, ser. 2, vol. 9, p. 410.

with similar expressions for B, C, and D. Let also

$$\Sigma F_{ijkl} x_i x_j x_k x_l = 0 \qquad (i \leqq j \leqq k \leqq l \leqq 4)$$

be the equation of the same surface in the ordinary form.

Equating the coefficients of like terms in the two equations of the surface, we obtain the following thirty-five equations:

$$\Sigma a_{ii}^2 = F_{iiii} \qquad (i = 1, 2, 3, 4),$$

$$\Sigma 2(a_{ii} a_{ij}) = F_{iiij} \quad (i, j = 1, 2, 3, 4; i \neq j),$$

$$\Sigma (2 a_{ij} a_{jj} + a_{ij}^2) = F_{iijj} \quad (i, j = 1, 2, 3, 4; i \neq j),$$

$$\Sigma (2 a_{ii} a_{jk} + 2 a_{ij} a_{ik}) = F_{iijk} \quad (i, j, k = 1, 2, 3, 4; i \neq j \neq k),$$

$$\Sigma 2(a_{12} a_{34} + a_{13} a_{24} + a_{14} a_{23}) = F_{1234},$$

the summation extending over the quantities b, c, and d.

The necessary and sufficient condition that (1) be the equation of a general quartic surface is that these quantities F_{ijkl} be independent functions of a_{ij}, b_{ij}, etc. They will be independent functions of these quantities if, and only if, at least one determinant of order thirty-five of the Jacobian matrix of thirty-five rows and forty columns

$$\begin{vmatrix} \partial F_{1111}/\partial a_{11} & \partial F_{1111}/\partial a_{12} & \cdots & \partial F_{1111}/\partial d_{34} \\ \partial F_{1112}/\partial a_{11} & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \partial F_{1234}/\partial a_{11} & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \partial F_{1234}/\partial d_{34} \end{vmatrix}$$

be different from zero.

If any of these determinants is different from zero, it will be impossible to find thirty-five quantities, $y_1, y_2, \cdots, y_{35}$, satisfying simultaneously the forty equations

$$\frac{\partial F_{1111}}{\partial a_{11}} \cdot y_1 + \frac{\partial F_{1112}}{\partial a_{11}} \cdot y_2 + \cdots + \frac{\partial F_{1234}}{\partial a_{11}} \cdot y_{35} = 0,$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\frac{\partial F_{1111}}{\partial d_{34}} \cdot y_1 + \frac{\partial F_{1112}}{\partial d_{34}} y_2 + \cdots + \frac{\partial F_{1234}}{\partial d_{34}} \cdot y_{35} = 0.$$

Consider, now, the system of quadrics

$$\alpha A + \beta B + \gamma C + \delta D = 0.$$

In this system there are, in general, ten* which break up into pairs of planes. Let the coordinate planes $x_1 = 0$ and $x_2 = 0$ be taken to coincide with one of these pairs of planes and $x_3 = 0$, $x_4 = 0$ with another pair. We may then, in the above forty equations, put $c_{12} = 1$ and the remaining $c_{ij} = 0$, also $d_{34} = 1$ and the remaining $d_{ij} = 0$.

It is now necessary that $y_{26} = y_{31} = y_{27} = y_{34} = y_{22} = y_5 = y_{25} = y_8 = y_{28} = y_{29} = y_{18} = y_{32} = y_{16} = y_{17} = y_{35} = y_{23} = y_{30} = y_{24} = y_{33} = 0$. Twenty-four of the equations are therefore satisfied. In order that the other sixteen be satisfied by quantities y not all zero, it is necessary and sufficient that the determinant

$$\begin{vmatrix}
a_{11} & b_{11} & & & & & & & & & & & & & & \\
& & a_{22} & b_{22} & & & & & & & & & & & & \\
& & & & a_{33} & b_{33} & & & & & & & & & & \\
& & & & & & a_{44} & b_{44} & & & & & & & & \\
a_{13} & b_{13} & & & & & & & a_{11} & b_{11} & & & & & & \\
a_{14} & b_{14} & & & & & & & & & a_{11} & b_{11} & & & & \\
& & a_{23} & b_{23} & & & & & & & & & a_{22} & b_{22} & & \\
& & a_{24} & b_{24} & & & & & & & & & & & a_{22} & b_{22} \\
& & & & a_{13} & b_{13} & & & a_{33} & b_{33} & & & & & & \\
& & & & a_{23} & b_{23} & & & & & & & a_{33} & b_{33} & & \\
& & & & & & a_{14} & b_{14} & & & a_{44} & b_{44} & & & & \\
& & & & & & a_{24} & b_{24} & & & & & & & a_{44} & b_{44} \\
a_{33} & b_{33} & & & a_{11} & b_{11} & & & a_{13} & b_{13} & & & & & & \\
a_{44} & b_{44} & & & & & a_{11} & b_{11} & & & a_{14} & b_{14} & & & & \\
& & a_{33} & b_{33} & a_{22} & b_{22} & & & & & & & a_{23} & b_{23} & & \\
& & a_{44} & b_{44} & & & a_{22} & b_{22} & & & & & & & a_{24} & b_{24}
\end{vmatrix}$$

be equal to zero. Denote the columns of this determinant in order by (1), (2), etc. Replace the first column by b_{11} (1) $- a_{11}$ (2) $+ b_{22}$ (3) $- a_{22}$ (4) $+ b_{33}$ (5) $- a_{33}$ (6) $+ b_{44}$ (7) $- a_{44}$ (8) $+ b_{13}$ (9) $- a_{13}$ (10) $+ b_{14}$ (11) $- a_{14}$ (12) $+ b_{23}$ (13) $- a_{23}$ (14) $+ b_{24}$ (15) $- a_{24}$ (16). The first column so obtained vanishes identically. The determinant therefore vanishes identically and the F_{ijkl} are not independent. The surface $\phi_2(A, B, C, D) = 0$ is not, therefore, the most general quartic.

In order that the equation of a quartic may be put in the above form, only a single condition must be satisfied by the coefficients. For if we omit F_{1234}, and consider the remaining F_{ijkl} as functions of the a_{ij}, etc., the Jacobian matrix does not vanish identically. This may be seen by substituting the particular values $a_{11} = b_{22} = c_{33} = d_{44} = 1$, the remaining a_{ij}, etc., being zero. Hence these quantities F_{ijkl} are independent.

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* Reye, *Crelle*, vol. 82, p. 76; vol. 86, p. 89.

Finally, consider the quartic

$$\theta_2(A, B, C, D, E) = 0.$$

Let this equation be reduced to

$$A^2 + B^2 + C^2 + D^2 + E^2 = 0.$$

Equate the coefficients of the terms of this equation to the corresponding terms of

$$\Sigma k_{ijkl}\, x_i x_j x_k x_l = 0 \qquad (i \leqq j \leqq k \leqq l \leqq 4)$$

and determine the jacobian matrix as before. That the determinants of this matrix do not all vanish identically is seen by taking for A, B, etc., the particular expressions

$$A \equiv x_1^2,\ \ B \equiv x_2^2,\ \ C \equiv x_3^2,\ \ D \equiv x_4^2,\ \ E \equiv x_1 x_2.$$

Since these determinants do not vanish identically the k_{ijkl} are independent. Hence *the equation of an arbitrary quartic surface can be put into the form*

$$A^2 + B^2 + C^2 + D^2 + E^2 = 0.$$

URBANA, ILL.,
September, 1907.

SYMBOLIC LOGIC.

L'Algèbre de la Logique. Par LOUIS COUTURAT. Collection Scientia, No. 24. Gauthier-Villars, Paris, 1905. 100 pp.

Symbolic Logic and its Applications. By HUGH MACCOLL. Longmans, Green, and Co., London, 1906. xi + 141 pp.

The Development of Symbolic Logic; a Critical-Historical Study of the Logical Calculus. By A. T. SHEARMAN. Williams and Norgate, London, 1906. xi + 242 pp.

SYMBOLIC logic is in the interesting though somewhat precarious state of being little known, less used, and much scorned by the majority of mathematicians and philosophers, for whom it might supposedly offer a region of intimate contact and mutual admiration. Meanwhile it has its own ardent supporters whose proselytism is at times almost as fanatical and extravagant as it is unavailing. One is tempted to draw a parallel with the subject of quaternions, which was offered insistently but ineffectively at bargain prices to both mathematicians and physicists until the nearly simultaneous death of Tait and Joly caused a temporary and perhaps permanent lull in the propaganda.

How far the analogy may be carried cannot yet be determined; for neither quaternions nor symbolic logic has run its course. But even if we admit that quaternions can never realize the early hopes which were entertained of it, we cannot be blind to the fact that now, with its consort of grassmannian origin, it stands at the head of a large family of investigations which has allianced itself with the theories of groups and of matrices and which through vector analysis has actually found employment in the mathematical encyclopedia. It would perhaps be rash to forecast the future of symbolic logic by any too close translation of what has occurred in another field; but the attempt to look somewhat into the future has an irresistible lure. To get on safer ground and to reach a point from which the setting of the three books under review may be seen to better advantage, it will be well to run briefly through the history of symbolic logic.

The rudiments of a symbolic method in logic may be seen in Euler's diagrams. For the algebra of classes or of propositional functions they offer a ready expression of the relation of inclusion, and the inequality of mathematics is so vividly suggested that no surprise should be felt if the discovery were made that Euler had experimented with truly symbolic methods. Certain it is that Leibniz perceived the possibility and apparently believed in the desirability of the introduction of symbolism into logical doctrine. It was Boole, however, who first made public an extended treatment of logic in symbolic terms. It should not be overlooked that his point of view was to treat logic by mathematical methods; not mathematics by purely logical methods.

Following Boole, there was a considerable body of work done by those primarily interested in logic. We may mention Charles Peirce, Mrs. Ladd-Franklin, MacColl, and Schröder. Aside from affording a formulation of the simpler laws and processes of logical procedure, these investigations disclosed the possibility of constructing a boundless extension of formal or algebraic logic by mere manipulation of the formulas. In fact it was possible to go on and on in logical algebra just as in ordinary algebra. The huge structure thus raised in all its intricacies and complications must naturally have an even less vital bearing on the fundamental problems of logic than similar refinements of pure mathematics have in the discussion of the physical world. There would be the same fruitfulness and the same barrenness.

It was at this point that Frege and Peano came forward with essentially new ideas — Frege in the early eighties and Peano in the late eighties of the last century. Unfortunately Frege's highly complicated symbolism prevented his keen philosophical reasoning from receiving much attention until very recently, and Peano's symbolism would very likely have proved a similar obstacle had his methods not appealed so strongly to his pupils as to build up a large school whose combined results naturally forced themselves on public attention. The new ideas were simply that mathematics was but a branch of logic, of symbolic logic. As early as 1889 Peano was very insistent on the fact that numbers and points were mere symbols. This was a great step to make.

And yet, when once made, it could hardly fail to appear inevitably suggested by what had already been accomplished. The well known geometries of Lobachevsky and Riemann were evidence enough that it was not our physical space but the system of axioms which afforded the logical foundation of geometry. And the geometric principles of correspondence, whether the simple dualities in plane and space or the more complicated transformations such as the line-sphere transformation of Lie, showed conclusively that it was not the undefined symbols and axioms but their interpretation which made the visual geometry. Thus on both sides the material for making the generalization lay right at hand. This new point of view brought to symbolic logic a new vitality, much as the discovery of a new relation between mathematics and physics has always brought fresh energy to mathematical investigations. The early point of view of Boole had now been changed to its opposite.

This change was far more vital than might at first be imagined. It emancipated logic from the typical form of reasoning found in the statements: All liberals are free-traders; X is a liberal; hence X is a free-trader. If logic is to handle all mathematics, it must deal successfully with the null class and the infinite class. The older logic was too vague, too full of tacit assumptions to stand such a strain. The researches of Cantor on transfinite numbers would alone show that the little word "all" had to be handled very gingerly. The wider field which was opened was not for symbolic logic merely, but for general logic as well. If the sufficiency of a definition or concept is to be tested, there is small value in selecting the simplest case as a test stone; the most complicated should be sought out, and Cantor's work has furnished the necessary complications for considerably clarifying, enlarging, and sharpening up our fundamental logical assumptions. The process is not yet finished.

Some of the advance which has been made towards a more perfect logic has been done without the use of what is strictly symbolic logic. The time may come when symbolism may be entirely discarded for such purposes and when symbolic logic as such may be a thing of the past. It should not then be forgotten, however, that those who first started upon the advance introduced an improved symbolic logic to help their progress, which without it would have been much slower and less effective. At present there remains much to accomplish which apparently cannot be done effectively without some acquaintance with symbolic methods. But it is not easy to admit that such a complete use of formalism as is advocated and illustrated by Peano in his Formulario can ever reach general acceptance. If it could, it may be true that a volume the size of the Jahrbuch might contain practically all the mathematical investigation of a year instead of merely short abstracts. But it is a recognized psychological fact that although condensation and abstraction up to a certain point aid the mind in grasping a result, yet after that point they render the task more difficult on account of the fatigue incident to excessive attention. The problem is essentially one in minimum fatigue, and notwithstanding the experiments on mental and physical dieting we shall probably as a majority stick to the feeling that we all need some hay with our grain.

Couturat's Algebra of logic aims to supply in brief and readily accessible form the elementary principles of the subject developed from the algebraic point of view rather than from the logical. It thus gives the mathematician an example of a non-quantitative algebra and the logician a chance to become acquainted with formal methods under auspicious circumstances. The terms and symbols are carefully explained and illustrated, and the algebraic work is for the most part maintained on a very simple and elementary level. The essentials only are given. For the learner this is a great advantage as compared with the detailed exposition of a large treatise like Schröder's. The logician will doubtless regret that the author did not give an article or two to the discussion of the different canonical types of syllogism, and indeed it might have been well to put in the text at an early point the elegant inconsistent triad of Mrs. Ladd-Franklin with a few remarks on its application. As it is, the author remains true to his intention of presenting the algebra of the subject without especial reference to the classical logic.

The first twenty articles give the reader a statement of the postulates and their immediate consequences. The matter is so presented as to admit of the dual interpretation conceptual and propositional, and these interpretations are constantly formulated in words. The postulates are chosen from the point of view which appears to the author as that most nearly in coincidence with the usual logical interpretation, and are not subjected to the critical examination which would be necessary if the chief interest were to set up a complete system of independent postulates. This is, of course, very much to the advantage of the ordinary reader who would merely be wearied by such refinements. On the other hand the author takes pains to make such comments as will inform the careful reader of many of the refinements which are really essential to a correct point of view. For instance on page 15, after showing how inclusions may be transformed into equalities, he states that if the relation of equality instead of the relation of inclusion had been taken as an undefined symbol, then the relation of inclusion could have been defined and the principle of the syllogism could have been proved. This statement contains the meat of what is perhaps the most fundamental idea in the whole subject, namely, that the definitions and theorems depend on the undefined symbols and postulates, which any author may assume with considerable arbitrariness, so that what is a theorem or definition for one author may be an undefined symbol or postulate for another, and vice versa. The presentation of the troublesome 0 and 1 (all or true) and of negation is such as to leave little room for doubt or misunderstanding. In explaining the law of duality, the distinction between primary and secondary propositions is insisted upon.

Articles 21 to 55 inclusive are of a different nature; they deal more especially with the logical algebra and contain numerous formulas which do not admit such an immediate logical interpretation as those of the preceding articles. The development of logical functions, the solution of equations in one or more unknowns, and the problem of elimination are among the subjects which might appeal most strongly to the mathematicians as affording differences and analogies to the corresponding problems of ordinary algebra. The laws of consequences and causes and the logical machine of Jevons would very likely be more of interest to the logician proper. Throughout these developments the author keeps the manipulation of the symbolism as simple as possible; but undoubtedly it would still offer obstacles to those quite unused to algebraic methods.

The concluding articles of the book are given over to the discussion of formulas which arise only in the calculus of propositions. As is known, it is only to a certain extent that the duality between propositions and classes may be carried. The principle of assertion introduces a fundamental difference. The postulate $(a = 1) = a$, which expresses this principle, enables a considerable reduction in the general formulas to be carried out — in short the calculus of propositions reduces to the calculus of classes where the universe of discourse has but one element.

In conclusion Couturat makes the point that this algebra of logic which he has set forth is of the nature of a formalization and extension of classical logic without going on to questions essentially different from those which have been treated since the time of Aristotle, namely, the relations of inclusion and implication, whereas a more general theory of relations seems necessary to a complete symbolic logic which shall suffice for the purposes of all mathematics.

From what has been said it will undoubtedly appear clear that Couturat has supplied a book which in no way duplicates anything in logical literature and which furnishes an easy and thorough introduction to the larger treatises. At the same time it is sufficiently advanced to cover more than the amount of symbolic logic which may be required by one not primarily interested in the subject. The author has had on hand for some time the preparation of a general treatise on logistics in which we may expect to find a completer discussion of symbolic logic as it now is.

For thirty-five years MacColl's name has been associated with the advance and with the criticism of symbolic logic. Aside from his numerous publications in various periodicals, he has given a somewhat lengthy account of his system in the third volume of the reports of the congress of philosophy at Paris in 1900. His present publication in book form appears to be an amplification of that account. One of his chief aims, as he states, has been to adopt notations which shall lead to a simple discussion of problems of probability and limits. It should be remarked that he uses limit in the sense in which it is used in the expression "limits of an integral" and not as a constant approached by a variable. For his purposes MacColl has seen fit to introduce a notation entirely different from any we have seen elsewhere. There is no suggestion, so far as the mere symbolism goes, of the relations of inclusion, whether subsumption or implication. A implies B is written $A:B$ and the statement that if A belongs to class B then C belongs to class D is formulated as $A^B : C^D$.

It may be interesting to go somewhat into detail as to the author's notation. He has five general exponents, $\tau, \iota, \epsilon, \eta, \theta$, which are used respectively to denote that the proposition to which they may be affixed is true, false, certain, impossible, variable. The difference between true and certain is this: A^τ means that A is true in some particular instance, whereas A^ϵ means that it is always true, that is, of probability 1. Again A^ι means that A is false in some particular instance, whereas A^η means that A contradicts some datum or definition, that its probability is 0. To quote further we find that "A^θ asserts that A is neither impossible nor certain, that is, that A is possible but uncertain. In other words, A^θ asserts that the probability of A is neither 0 nor 1, but some proper fraction between the two." As the author apparently does not define probability our interpretation of what he means by it must belong to class θ.

According to the generally accepted definition, we may agree that the probability of a certainty is 1 and of an impossibility is 0; but unless the class of objects is finite we cannot agree that the statements "possible but uncertain" and "having a probability which is a proper fraction" are identical. For suppose we mark off an interval of length unity on a line and let us further mark the middle point of the interval with the letter M and let P represent any point of the interval. Then the proposition P is M appears to be neither certain nor impossible but variable and so with the proposition P is not M. Yet in the former case the probability would appear to be 0 and in the latter case to be 1. Whether the author has been insufficiently careful in his statement concerning A^θ or not, the fact remains that probability is a dangerous thing and might better be left out. As a mathematician, one would hesitate a long while before he was willing to commit himself to a definition of probability which could be applied to a proposition derived from a propositional function which had an infinite number of applications in the universe of discourse.

The first six chapters of MacColl's book are used to set up his logical system and develop his analysis. It would take an expert a long time to decide which of the many statements the author intends to have regarded as postulates or axioms. It is probable that his method of procedure is not all in sympathy with the recent postulational tendency and that he reserves the right to introduce any statement at any time without making much ado over it, provided only that it appears to him to be correct. In this he is quite in touch with the everyday logician. Yet on the whole he is explicit in his statements and not readily misunderstood; it is sometimes difficult to understand him at all. For instance, there is the long standing dispute between him and some other logicians as to whether a proposition may be sometimes true and sometimes false. This seems to be largely a matter of definition. As MacColl's definition of proposition does not differ much from the usual definition of propositional function, we may be justified in concluding with some of his milder opponents that according to his own definitions he is right.

Chapters VII to XIII inclusive treat the usual subjects of classical logic. The treatment is careful and critical and affords a large amount of material for any one interested in the logical side of symbolic logic. On page 47 is found the startling statement that not one syllogism of the traditional logic is valid in the form in which it is usually presented in our textbooks. The author merely wishes to emphasize the fact that "Every A is B; every B is C; therefore every A is C" is incorrect whereas "If every *A* is *B* and if every *B* is *C*, then every *A* is *C*" is correct. And he is perfectly right in emphasizing it. The logician might attempt to wriggle out of the difficulty in one of two ways: 1° by saying that he meant the if's and the then instead of the assertions and the therefore, or 2° by claiming that it was always assumed that the asserted premises were true. In the former case we could certainly reply that it was his business to say what he meant, and in the latter case we should point out that the useful principle of reductio ad absurdum depends on drawing valid inferences from hypothetical or else erroneous premises.

The author next takes up the nineteen classical syllogisms as revised by the insertion of if's and then's. Four of these types, namely, Darapti, Felapton, Bramantip, Fesapo, have generally been held by symbolic logicians to be incorrect forms of inference or impure syllogisms. It is certain that they do not satisfy the inconsistent triad of Mrs. Ladd-Franklin and that they are the only ones that do not. Darapti reads as follows: If every *Y* is *Z* and every *Y* is *X*, then some *X* is *Z*. The difficulty with the triad in this case is that if no *Y* exists there is no way of connecting *X* and *Z*, and most symbolic logicians have been willing to add $Y \neq 0$ to make the inference sure.* Of course if it be admitted that every *Y* implies the existence of at least one *Y* or that some *X* does not imply the existence of any *X*, there is no difficulty. There remains therefore only the case where *Y* is null and where some is taken to imply existence. MacColl claims that the inference is sure in this case; and a large number of logicians would probably agree with him. Perhaps they are right; but part of MacColl's reasoning looks suspiciously as if it could be applied equally well to cases which everybody would admit were false.

After offering so much in disapprobation of the book under review, it is a great pleasure to be able heartily to approve of the manner in which the author treats the so-called canons of syllogistic validity. These are: 1) Every syllogism has three and only three terms. 2) Every syllogism consists of only three propositions. 3) The middle term must be distributed at least once in the premises ; and it must not be ambiguous.* 4) No term must be distributed in the conclusion unless it is also distributed in one of the premises. 5) We can infer nothing from two negative premises. 6) If one premise be negative, the conclusion must be so also ; and conversely. The author proceeds to demolish this structure. Let us take a simple example in connection with the fifth canon. Suppose the premises : No X is Y, no Y is Z (the if's as usual are omitted). Can nothing be inferred ? It is unnecessary to wake up the old-school logician for such a matter. Canon 5) is obviously sufficient. Suppose by an extremely artificial rearrangement of language we write : All Y is not-X, all Y is not-Z. Can nothing be inferred ? A foolish question : this is obviously the valid Darapti and some not-X is not-Y!

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* It may be of interest to quote Schroeder's example. We omit the if's because the premises are so stated as to be incontestable. Every rectangular equilateral triangle is rectangular; every rectangular equilateral triangle is equilateral; hence some rectangular triangle is equilateral. The conclusion is true in spherical geometry where there are equilateral rectangular triangles but untrue in the plane where there are none.

It is by such chicanery as this that the logicians of the old school persist in saving the face of their canons and at the same time render the study of their logic distasteful to many and disgusting to those who are used to drawing conclusions rapidly and accurately. The author is quite justified in maintaining that his formulas are more reasonable. If one is willing to discard the four disputed cases, the inconsistent triad is even more convenient. The statements some X is Y and some Y is X are obviously equivalent and may be written $XY \neq 0$. The statements no X is Y and no Y is X are written $XY = 0$. The statement all X is Y is equivalent to no X is not-Y and is written $XY' = 0$. The triad may then be written

$$(XY = 0)(Y'Z = 0)(XZ \neq 0) \text{ is impossible,}$$

$$\text{or } (XY = 0)(Y'Z = 0) \text{ implies } (XZ = 0), \text{ universal syllogism,}$$

$$\text{or } (XY = 0)(XZ \neq 0) \text{ implies } (Y'Z \neq 0), \text{ particular syllogism.}$$

It takes less time to acquire a ready working knowledge of this rule than to learn the poem about Barbara, to say nothing of learning how to apply it.

The author next goes on to the question of supplying the missing premise in an enthymeme and to similar questions. He then takes up some disputed points such as the alleged duality between classes and inclusion on the one hand and propositions and implication on the other. MacColl was one of the first to argue that the duality is by no means complete, and by this time his contention is pretty well established despite the fact that he himself is not careful in drawing a distinction between propositions and propositional functions. Another matter in which the author has always been very active is the discussion of the null class and the paradoxes which arise from its introduction into the logical system. That there are difficulties in the theory of the null class and that these difficulties, and those connected with the class of all, perhaps more than anything else stand in the way of the ordinary logicians when they attempt to follow the symbolists no one will deny. The solution which the author gives by introducing a theory of unrealities does not, however, appear any more satisfactory to most symbolists than their theory appears to him.

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* It is extraordinary that anybody should think it worth while to canonize this latter statement !

Two very useful chapters are those on the solutions of some questions taken from recent examination papers and on the definitions of technical terms. One of the difficulties which a reader who is not especially trained in the nomenclature of logic experiences is the sudden appearance of some term such as modus tollendo ponens or complex constructive dilemma, and so on. The author's systematic tabulation of definitions of technical terms is therefore a great convenience. It is impossible, however, to pass over one of the author's definitions without a word of protest. He says: "Much confusion of ideas is caused by the fact that each of these words (infinite and infinitesimal) is used in different senses by mathematicians. Hence arise most of the strange and inadmissible paradoxes of the various non-euclidean geometries. To avoid all ambiguities I will define. . . . The symbol α denotes any positive quantity or ratio too large to be expressible in any recognized notation and any such ratio is called a positive infinity." The first sentence is correct although the confusion as far as mathematicians is concerned is not insuperably great. The second, about non-euclidean geometries, appears to be a bit of news. The third is apparently both ambiguous and unintelligible. Is the α beyond the alephs and omegas? As a matter of fact the α does not appear to be anywhere in particular. And this was in 1906.

The last five chapters, XIV to XVIII, deal with the calculus of limits. It is this portion of the work which the author might expect to find most fully reviewed here. But the mention will only be short. The problem is to find those values of the variable for which certain expressions remain positive or become zero or remain negative, and more especially to determining the limits of integration for multiple integrals. The question of the limits of integrals was treated some long time since by the author in the *Proceedings of the London Mathematical Society*. He has also contributed other mathematical articles to that periodical and to the Mathematical questions and solutions from the *Educational Times*. The great advantage of his method is, as he states, that it is independent of diagrams and is equally applicable when the number of variables is great, whereas diagrams become complicated in three dimensions and impossible in a higher number. But like the old-school logicians, most mathematicians have an obstinate fondness for their own methods and will probably struggle on with them the best they can.

As regards the numerous points still in dispute between the author and other logicians one further word should be said. The disputations have been largely controversial. In some cases MacColl has won and in some he has lost; and a similar division of the spoils is not unlikely in the future. The unfortunate thing is that there has been so much controversy and so little proof whether on one side or on the other. In fact the mass of verbiage and illustration is often quite bewildering; so that the reader of the communications pro and con is unable to decide on anything. If in such cases as these the disputants would only refrain as much as possible from talking and devote their energies to setting up sets of postulates as definitely as they could and then developing their consequences with reference to alternative systems, some distinct and intelligible results might be obtained in short order. These results might be largely negative in that they showed that different logical systems led to similar conclusions with regard to all points not in dispute and to different conclusions with respects to points in dispute; but even that much conclusively shown would amount to a great deal in clearing the atmosphere. In other words why, pray, do not logicians use to a greater extent the refinements of deductive logic when they are discussing delicate points; mathematicians have adopted that point of view in dealing with geometries. Evidently the doctors do not take their own medicine.

Shearman, if his text be taken in evidence, is precisely one of these controversial and dialectic logicians. His book, being in itself a history and a criticism of symbolic logic, is particularly hard to criticize in detail. He has much more of interest to offer to logicians, and especially to such as stand aloof from symbolic logic, than to mathematicians. For instance, he starts his introduction with the statement that it is his object to show that during the last fifty years there has been a definite advance made in symbolic logic. As there was very little symbolic logic in 1856 no one will dispute his contention, and as far as mathematicians are concerned one may say that the chief advance in symbolic logic has been made in the last twenty years. And this work is left almost untouched by the author. As his point of view is partly historical and partly critical, and as there is no very sharp distinction made between the two in the text, it is at times practically impossible to decide whether any particular series of statements is to be interpreted as a quotation from past reasoning which led to certain conclusions or as an argument of the author's.

The first chapter deals with the question whether letters should be taken to represent classes or propositions. After thirteen or fourteen pages of dialectic we reach this conclusion: "I think, then, that it is correct to hold that our letters may represent either classes or propositions, but that we must be careful to notice that the rules to be adopted in working out problems are not the same in the two cases. This view has been adopted by Boole, Venn, and Schröder, who, so far as the point in question is concerned, are thus seen to have been proceeding in the right direction." This is typical of the author's critical-historical conclusions, and the dialectic which goes before is typical of his analysis. To one at all accustomed to the use of symbols it might seem obvious that letters may be used to represent either classes or propositions or anything else, provided that the proper rules were in each case observed; but the author has seen fit to hedge a bit in a footnote, where he states that his argument has reference to the earlier problems of symbolic logic and that "when we come to deal with problems that are not included within the scope of the Boolian treatment, I admit that it is better to let symbols stand primarily for propositions."

This sort of elaborate treatment of questions of notation, this hedging about every statement with a qualification, this mixture of criticism and history to a point where the reader cannot tell what is meant, this formulating of conclusions in the first person with the author's own position as a final test for correctness and rightness abound throughout the book. They are not peculiar to Shearman alone, but permeate the work of a large number of logicians who apply the adjective symbolic to their work, although they rarely if ever do the smallest piece of research with truly symbolic methods. What real justification they have in applying the term to themselves is merely a matter of conjecture. Perhaps they do not know what other term to use. They certainly are not classical logicians, because they have been so bold as to see that reading Attic Greek is not a sufficient condition for becoming a complete logician, and they are certainly not up-to-date symbolic logicians, because they have not perceived that reading modern Italian is a necessary condition. Notwithstanding, they attack each other with ardor, assault a really conscientious symbolist like MacColl or Mrs. Ladd-Franklin with virulence, and freely distribute their willing or unwilling applause to such authorities as Boole, Schroeder, Peano, Frege, Couturat, and Russell. Although they are thus proficient in controversy, it may be doubted if they have any even embryonic notion of what really constitutes a proof, a definition, or a postulate.

Shearman's second chapter is on the symbols of operation. He states on page 34 that most writers on the subject of symbolic logic have undoubtedly introduced symbols of operation. It is interesting to note the implication that one may perfectly feasibly construct a symbolic logic without any symbols of operation. This is but corroborative of our statement that many logicians who term themselves symbolic eschew symbolism to a very great extent. It would appear that they might better use the adjective formal in place of symbolic. For any logic which is other than an investigation in the psychology of the mental processes of reaching conclusions (as Wundt's Logik, for instance) is bound to be symbolic to a certain extent. The mere employment of language at all is symbolism. To go a little further the use of *X*'s and *Y*'s and *Z*'s in stating the syllogisms is symbolic. To this extent all formal logic is symbolic. But the real essence of symbolic logic as contrasted with merely formal logic is its recognition of the value of a preponderating use of symbols as against language of the ordinary sort on account of the greater precision, the greater abstraction, and the readier manipulation which are thereby obtained.

Although a large amount of the author's second chapter will appear either trite or misguided to those acquainted with symbolism, it may be of value to others; and some of the discussions, such as that connected with the difference between exclusive and non-exclusive alternatives, are illuminating to a larger circle of readers. The next chapter on the process of solution gives an idea of solution whether analytic or diagrammatic. The presentation would gain considerably in clearness if the author had inserted at least one diagram and had outlined the analytic method with more analytic detail. The next chapter discusses the question of the extensive versus the intensive basis for the calculus of logic. The treatment is mainly carried on as a critique of Castillon's system, which was based on intension. This is particularly interesting inasmuch as Castillon's logic is comparatively little known. The author inclines very strongly to the opinion that the basis should be extensive — in fact he is by no means completely in sympathy with Russell's idea that it is in a domain intermediate between pure intension and pure extension that symbolic logic has its lair. But just what the author would do in handling the infinite classes does not appear. He avoids the very subject which has been mainly responsible for the recent advances in symbolic logic.

Shearman spends chapter V in trying to show that Jevons and MacColl have not contributed much to symbolic logic. Jevons's deficiency was due to his lack "in power of originating important logical generalizations and his failure to appreciate the full significance of the work done by other logicians." This is perhaps true; but the statement is so extreme that it would apply with equal or greater force to so many others as to nullify its value as a criticism of Jevons in particular. MacColl's difficulty is his lack of cooperation with other symbolists. The words about Jevons are few; but more than twenty pages are devoted to reducing MacColl to an infinitesimal. The length of the argument shows one thing very conclusively, namely, MacColl's prominent position in symbolic logic during the last thirty years. As has been stated above, the points in dispute with MacColl are partly matters of definition in which any author is more or less free to use his own discretion, partly matters of notation in which an author is bound by little more than his regard for the ease of his readers, partly matters connected with much mooted questions which are not yet settled entirely to the satisfaction of any one class of logicians, to say nothing of all classes together.

Until this triplicate division of the dispute is more carefully observed and until more refinements have been employed than are available in a mere argument, we cannot agree with the author in stating that the controversies have been settled, even if a good deal of light has been thrown on them.

Chapter VI is on the later logical doctrines, namely, the doctrine of multiple quantification, the logic of relatives, and the new symbolic logic. The first will be passed without comment. In treating the logic of relatives, the author appears to be unacquainted with Russell's or Royce's investigations on the subject, and without this any present discussion of relatives is to a large extent lacking in completeness. This defect is partly remedied in the treatment of the new symbolic logic. Here there is some account of the work of Frege, Peano, Russell, and Whitehead. This will probably afford interesting reading for logicians rather than for mathematicians, who will prefer to go to the original sources instead of relying on what is necessarily an inadequate account. In the concluding chapter on the utility of symbolic logic, the author lapses into a metaphysics or something similar that apparently does not advance him very far toward showing that symbolic logic has any particular utility.

In concluding with this critical-historical study of the logical calculus, it should be stated that the mathematician must not turn to it in expectation of finding what he is most likely to want. There is nothing in it which will throw light on the nature of a defintion, a postulate, or a proof, or of what constitutes a deductive system; little or nothing which will suggest the existence of reasoning other than syllogistic; and not much more which would indicate the analytic elegance of the threefold calculus of propositions, classes, and relations. The mathematician is almost certain to find more of value to him in the excellent though brief account of symbolic logic by E. V. Huntington and Mrs. Ladd-Franklin in the ninth volume of the Encyclopedia Americana. It would be difficult to cite a better reference.* Logicians may find in Shearman's book much that is interesting and valuable to them; but as a mathematician writing in a mathematical journal, it is impossible to refrain from cautioning them against imagining that they are becoming acquainted with symbolic logic, or the deductive system in general, as mathematicians know it and use it. It may be hoped that logicians, too, will see fit to consult the article in the encyclopedia just mentioned or some other source of similar character, such as Pieri's inaugural address, before they permit themselves to form an opinion on the accomplishments, value, and recent advances of symbolic logic.

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* We should not omit to cite also the admirable and quite recent brochure entitled Uno sguardo al nuovo indirizzo logico-matematico delle scienze deduttive; discorso letto dal professore M. Pieri inaugurandosi l'anno accademico 1906–'07 nella Reale Università di Catania. Francesco Galati, Catania, 1907. 62 pp.

EDWIN BIDWELL WILSON.

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SHORTER NOTICES.

H. Durège, Elemente der Theorie der Funktionen einer komplexen veränderlichen Grösse, in fünfter Auflage, neu bearbeitet, von LUDWIG MAURER, mit 41 Figuren im Text. Leipzig, B. G. Teubner, 1906. 397 + x pp.

THE present work, although styled the fifth edition of Durège's well-known "Elements," is in reality a new treatise. The title and the short historical introduction have been retained ; aside from these, we have not remarked a trace of the original work. Nor could it be otherwise. The theory of functions has grown enormously since the days of Riemann. On the one hand, new fields have been opened up and explored, on the other the old tools of research have been given a greater refinement and many new ones have been added. The present author, in preparing a new edition, quite rightly decided not to patch up the old edifice, but to tear it down completely and erect a new one in harmony with the needs and tendencies of the present day. The result is an up-to-date treatise of moderate proportions, clearly and attractively written, which will surely have a widespread and well-deserved popularity.

The book starts out with an introductory chapter on real variables. Dedekind's theory of irrational numbers is sketched; such notions as simple and multiple limits, upper and lower limits, uniform convergence, also a few notions from the theory of point aggregates are briefly treated. The subject of integration is developed more fully and terminates with Gauss's relation between line and double integrals, which is later used to prove Cauchy's fundamental theorem.

After a short chapter on the arithmetic of complex numbers and their geometric representation, we begin on page 75 the study of the theory of functions of a complex variable. The definition of an analytic function chosen by the author, is Cauchy's, viz.: $f(z)$ shall have a continuous derivative. One hundred pages, divided into three chapters, suffice for the author to give the reader a tolerably well rounded survey of the general theory of one valued functions. The selection of topics is felicitous, the treatment is fresh and individual, and finally a happy mean is held between laxity and extreme rigor in the proofs.

The author has given Cauchy's integral relation

$$f(z) = \frac{1}{2\pi i}\int \frac{f(t)dt}{t-z}$$

a use we have not noticed elsewhere. He shows very readily that if f is a continuous function of x, y and satisfies this relation, it is analytic. By the aid of this simple property, he shows that a uniformly convergent series or product of analytic functions is analytic. In this connection the reviewer may be permitted to express his surprise that Herr Maurer has not made use of the theorem of page 118 relative to the termwise integration of a series, to shorten very much the proof of Taylor's and Laurent's theorems. With its aid the lengthy discussion of the remainders is unnecessary.

Now follows a chapter on doubly periodic functions. This break in the general theory serves a two-fold purpose. First, it gives the reader a chance to rest after a rather arduous march; secondly, he is given an opportunity to apply his new instruments to the study of an interesting and important class of functions. We know there is nothing to sustain a healthy interest like doing things.

We may be allowed to call attention to the pretty way the $\wp$ function is introduced. As simplest elliptic function, we choose one of order two. We take its pole double, as the order of its derivative is then three instead of four. For convenience we suppose this pole to be at the origin. The development of the function is thus

$$\frac{a_{-2}}{z^2} + a_0 + a_1 z + a_2 z^2 + \cdots$$

The choice of a_{-2} and a_0 completely fixes the function. We take now $a_{-2} = 1$ and $a_0 = 0$. It is then easy to show that this function is Weierstrass's $\wp$ function. Let us note in passing that the author has broken away from the traditional symbol $\wp$, which he replaces by p. The reviewer has never been able to understand the vogue of this fetish, unless it be that mathematicians, like children, take pleasure occasionally in some strange and outlandish novelty.

Leaving the elliptic functions, the reader is introduced to the theory of many valued functions. Here the notions analytic continuation, Riemann surfaces and Schwarz's principle of reflection are briefly treated. As an instructive example, the representation afforded by the elliptic function is studied in detail.

A noteworthy chapter on algebraic functions now follows. It would be difficult to imagine a better choice of material than that which compactly fills its fifty pages. Puiseux's cycles are determined and the corresponding Riemann surface is constructed. No attempt is made to reduce the general surface to one simply connected; the author has wisely restricted the discussion to the all important case where its branch points are of the first order. The chapter closes with a study of functions which are one valued on a given Riemann surface.

The last chapter, occupying just 100 pages, is devoted to the theory of homogeneous linear differential equations of the second order. This is an innovation in text-books on the function theory. We believe, however, it is one which will win many friends for the book.

The first problem is to determine the form of the solution in the vicinity of its singular points. The presence of certain singular points called points of indetermination causes great complication. Criteria are therefore established which characterize equations free from such points. They are the equations first studied by Fuchs and have been named in his honor. Simple examples of these are the equations satisfied by the hypergeometric function and the P function of Riemann. These are now studied in detail. When the independent variable describes a circuit about a branch point the fundamental integrals are subjected to a linear substitution. The coefficients of the latter are now determined, and the result finds at once an application in the study with Riemann and Schwarz of the representation afforded by the quotient of two independent solutions of the hypergeometric equation. We are thus led directly to the theory of automorphic functions. Space is found for a short discussion of the dreieck functions, *i. e.*, functions which map the half plane on a triangle whose sides are circles.

We hardly need to add in closing that we recommend Herr Maurer's book most heartily to the student. Even the instructor will, we doubt not, find fresh inspiration in perusing its pages, and find here and there a mode of treatment which he will be tempted to incorporate into his own lectures.

JAMES PIERPONT.

Elliptische Funktionen. Von HEINRICH BURKHARDT. Zweite, durchgesehene und verbesserte Auflage mit zahlreichen Figuren im Text. Leipzig, Veit and Company, 1906. 373 + xvi pp.

THE present edition is essentially a reproduction of the first, except that here and there a proof has been improved or a typographical error has been corrected. Numerous friends of the book have sent the author lists of errata. Thanks to their coöperation, the author hopes the formulas are now entirely reliable. This is certainly a most important feature in a subject which almost suffers from its inexhaustible wealth of formulas.

For a detailed account of the contents and tendencies of this superior work, the reader may consult an extended review in this BULLETIN for July, 1900, pages 452–463.

JAMES PIERPONT.

Quadratic Forms and their Classification by Means of Invariant Factors. By T. J. I'A. BROMWICH. Cambridge University Press, 1906. viii + 100 pp.

THE theory of elementary divisors, with which this book deals, is one of the most useful and perfect of algebraic theories. Although it is now nearly forty years since Weierstrass's fundamental paper was published, no treatment of the subject appeared in English until the year 1904, when a brief discussion was included in Mathews's revision of Scott's Determinants. This treatment is far from being suited to the needs of one wishing to penetrate for the first time into the theory, and the same is true of the only treatise on the subject which exists in any language, that of Muth. The appearance of Mr. Bromwich's book is therefore to be hailed with satisfaction as affording the first opportunity for those who prefer to use the English language to become familiar with this subject without an undue expenditure of labor.

The author replaces Weierstrass's term *elementary divisor*, which is now in universal use, by the term *invariant factor*. This is rather a high-handed proceeding and would seem to require justification. Weierstrass's term is a suitable one and admits of satisfactory translation into English; so that it is hard to see what the author's purpose can have been in making the change.

The author does not develop the theory except so far as it relates to the problem of the classification of quadratic forms; and, in fact, the whole book is directly connected with this problem, and with its geometrical interpretations in the cases of two, three, four, and to a less extent, five or six variables. The constant reference to geometrical illustrations, and the introduction of simple geometrical cases at an early stage form an admirable feature of the book. The author has elaborated a method of treatment, due in its essential features to Kronecker, which has the advantage over Weierstrass's original method of avoiding the necessity for the proof of certain difficult theorems concerning determinants.

In one respect the treatment is very complete since it covers not merely the cases originally considered by Weierstrass, but also the *singular case* first treated by Kronecker.

The method of presentation is thoroughly elementary, and a reader with some slight familiarity with the use of trilinear coördinates should have no difficulty in working through the little volume from cover to cover, and, in the end, really knowing what elementary divisors are and how they are to be used. After having said this, it may seem strange to add that the main results of the theory are not sufficiently emphasized, sometimes hardly even explicitly stated. This, however, is the case, and it makes the book of very much less value than it might otherwise have been to persons who wish to get at the main lines of the theory without going through all the details of the proofs and of the numerical examples which are more or less mixed in with them. The volume has no index.

MAXIME BÔCHER.

NOTES.

At the annual meeting of the London mathematical society held on November 14, the following officers were elected: W. Burnside, president; A. E. H. Love and J. H. Grace, secretaries; J. Larmor, treasurer; two vice-presidents and ten other members of the council. The following papers were read: By J. H. Maclagan-Wedderburn, "On hyper-complex numbers"; by T. J. I'A. Bromwich, "Addendum to a paper on the inversion of a repeated infinite integral"; by G. H. Hardy, "Generalization of a theorem in the theory of divergent series"; by W. H. Young, "Uniform and non-uniform convergence and divergence of a series and the distinction between right and left"; by J. E. Campbell, "Application of quaternions to the problem of the infinitesimal deformation of a surface"; by J. E. Wright, "Nodal cubics through eight given points"; by H. F. Baker, "Invariants of a binary quintic and the reality of its roots"; by E. W. Barnes, "On a transformation of hypergeometric series"; by M. J. M. Hill, "On a transformation of a certain hypergeometric series"; by J. E. Littlewood, "A general theorem] of integral functions of order less than one half."

The following parts of the Encyklopädie der mathematischen Wissenschaften are announced by Teubner of Leipzig to be in the press, and will probably appear during the winter: II, 1, A, 12, Allgemeine Reihenentwickelungen, by H. Burkhardt; II, 2, B, 2*a*, Arithmetische Theorie der algebraischen Funktionen, by K. Hensel; B, 3, Elliptische Funktionen, by J. Harkness and W. Wirtinger; III, 2, C, 4, Specielle ebene algebraische Kurven, by G. Kohn; C, 5, Allgemeine Theorie der höheren algebraischen Flächen, by G. Castelnuovo and F. Enriques; III, 3, D, 6*b*, Andere Differentialgleichungen der Geometrie, by C. Guichard; D, 7, Berührungstransformationen, by G. Scheffers; IV, 1_1, Heft 4; IV, 1_2, Heft 2; IV, 2_1, Heft 4; IV, 2_2, Heft 2; V, 1, Heft 4; V, 2, Heft 3; VI, 2, Heft 2.

Of the French edition of the encyclopædia, I, 1_3, I, 2_1, I, 3_2 and I, 4_2 are in the press.

The following additional works are in the press of Teubner; Führer durch die mathematische Literatur für Studierende, by F. Müller; Die Erfindung der Infinitesimalrechnung, by G.

KOWALEWSKI; Mathematical papers of the far east, by Y. MIKAMI; Grassmann's Werke, volume 3; Didaktik der Mathematik, by A. HÖFLER; Vorlesungen über die Weierstrasssche Theorie der irrationalen Zahlen, by N. v. DANTSCHER; Mengenlehre, II, by A. SCHOENFLIES; Theorie der algebraischen Zahlen, by K. HENSEL; Beiträge zur Theorie der linearen Transformationen, by W. SCHEIBNER; Grundzüge der Differential- und Integralrechnung, by G. KOWALEWSKI; Vorlesungen über Variationsrechnung, by O. BOLZA; Lehrbuch der Variationsrechnung, by C. CARATHÉODORY and E. ZERMELO; Algebraische Funktionen und ihre Integrale, by W. WIRTINGER; Vorlesungen über darstellende Geometrie I, by F. v. DALWIGK; Leitfaden der Kartenentwurfslehre, II, by K. ZÖPPRITZ; Vorlesungen über Geometrie, I, 1_2 (second edition), by A. CLEBSCH and F. LINDEMANN; Entwickelungen nach oszillierenden Funktionen (part six), by H. BURKHARDT; Geometrie der Kräfte, by H. E. TIMERDING.

AT the recent annual meeting of the royal society of London the royal medal was awarded to Dr. E. W. HOBSON, of Cambridge University, for his contributions to mathematics, and the Sylvester medal was awarded to Professor W. WIRTINGER, of the University of Vienna, for his researches in the theory of functions.

THE city of Faenza, Italy, in which TORRICELLI was born, will commemorate the three hundredth anniversary of his birth by the publication of his complete works. Professor J. VASSURA, of the royal lyceum at Forli, has been appointed chairman of the committee of publication.

PROFESSOR K. CARDA, of the technical school at Vienna, has been appointed to a full professorship of mathematics at the technical school at Prague.

PROFESSOR ST. JOLLES, of the technical school at Charlottenburg, has been promoted to a full professorship of mathematics.

PROFESSOR G. KOWALEWSKI, of the University of Bonn, has been appointed professor of mathematics at the academy of mines at Clausthal.

DR. M. LELIEUVRE has been appointed professor of rational mechanics at the scientific school at Rouen.

DR. G. HERGLOTZ, of the University of Göttingen, has been promoted to an associate professorship of mathematics.

DR. J. PLEMELJ, of the University of Vienna, has been appointed associate professor of mathematics at the University of Czernowitz.

DR. —. ROCQUEMONT has been appointed professor of applied mathematics at the scientific school at Rouen.

DR. —. GASSER has been appointed docent in geodesy at the technical school at Darmstadt.

DR. —. SALKOWSKI has been appointed docent in descriptive geometry at the technical school at Charlottenburg.

AT Williams College, Dr. C. S. ATCHISON and Mr. G. R. CLEMENTS have been appointed instructors in mathematics.

MR. G. E. CARSCALLEN has been appointed assistant in mathematics at the University of Illinois.

DR. GEORG SIDLER, emeritus professor of mathematics and astronomy at the University of Bern, died November 9, 1907, at the age of 77 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BERRY (W. J.). Differential equations of the first species. New York, Van Nostrand, 1907. 12mo. Cloth.

BOSMANS (H.). L'algèbre de J. Peletier du Mans départie en deux livres. Bruxelles, 1907. 8vo. 61 pp.

CAIN (W.). A brief course in the calculus. New York, Van Nostrand, 1907. 12mo. 7 + 280 pp. Half leather. $1.75

CHINI (M.). Lezioni di algebra, ad uso dei licei. Vol. I. Livorno, Giusti, 1907. 16mo. 8 + 166 pp. L. 1.80

EULER. Festschrift zur Feier des 200. Geburtstages Leonhard Eulers. Herausgegeben vom Vorstande der Berliner Mathematischen Gesellschaft. Leipzig, 1907. 8vo. 137 pp. M. 5.00

FALISSE (V.). Cours de géométrie analytique plane. 6e édition. Bruxelles, 1906. 8vo. 572 pp. Fr. 7.00

GOSIEWSKI (W.). Zasady rachunken prawdopodobienstwa. (Principes du calcul des probabilités). Warszawa, 1906. 8vo. 265 pp. M. 6.00

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THE FIRST REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE first regular meeting of the Southwestern Section of the AMERICAN MATHEMATICAL SOCIETY took place in St. Louis, Mo., on the 30th of November, 1907. About fifty persons, including the following members of the Society, were present:

Professor L. D. Ames, Mr. Charles Ammerman, Professor Florian Cajori, Professor A. S. Chessin, Professor C. E. Comstock, Brother Constantius, Professor S. C. Davisson, Professor G. R. Dean, Professor G. W. Droke, Dr. Otto Dunkel, Professor B. F. Finkel; Professor G. B. Halsted, Mr. W. W. Hart, Professor H. C. Harvey, Professor E. R. Hedrick, Dr. Louis Ingold, Dr. G. O. James, Professor O. D. Kellogg, Professor G. A. Miller, Professor H. B. Newson, Professor J. H. Scarborough, Professor J. B. Shaw, Mr. R. L. Short, Professor H. E. Slaught, Dr. Clara E. Smith, Mr. H. P. Stellwagen, Mr. E. H. Taylor, Professor C. A. Waldo, Professor B. M. Walker, Dr. Paul Wernicke.

The meeting was called to order by Professor Hedrick at 10:30 A. M., at the McKinley High School. The morning session adjourned at 12:30 P. M. The afternoon session was called to order at 3 P. M., at Washington University. The reading of papers was preceded by a short business meeting. Professor Chessin (chairman), Professor Kellogg (secretary) and Professor Newson were elected to serve on the program committee for the ensuing year. The University of Kansas was unanimously selected as the next meeting place of the Southwestern Section.

The following papers were read:

(1) Dr. P. WERNICKE: "On Euler's tactical '36 officers' problem."

(2) Dr. P. WERNICKE: "Extension of the map-color theorem to 3 dimensional space."

(3) Professor W. F. OSGOOD: "On the differentiation of definite integrals."

(4) Professor F. CAJORI: "Notes on the history of the slide rule."

(5) Dr. LOUIS INGOLD: "Note on areal cross ratios."

(6) Dr. G. O. JAMES: "A relation connecting aberration and parallax" (preliminary report).

(7) Professor J. B. SHAW: "A new graphical method for quaternions."

(8) Professor E. R. HEDRICK: "On a definition of the jacobian" (preliminary report).

(9) Professor A. S. CHESSIN: "On an integral appearing in photometry."

(10) Professor O. D. KELLOGG: "Real roots of an algebraic equation."

(11) Dr. LOUIS INGOLD: "Note on a connection between algebraic invariants and the invariants of a differential form."

(12) Professor H. B. NEWSON: "On the resultant of two collineations."

(13) Professor G. A. MILLER: "On the holomorph of the cyclic group of order p^n."

(14) Professor E. W. DAVIS: "Colored imaginaries on a cubic."

In the absence of the authors, Professor Osgood's paper was read by Professor Hedrick and Professor Davis's paper was read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Euler and Cayley have given some attention to the problem of arranging in a square of n rows and columns n^2 officers of n different regiments, the n of each regiment being of n different ranks. Every rank and regiment is to be represented in each row or column. Such arrays are possible unless $n \equiv 2 \pmod 4$. No proof is extant of their impossibility in this particular case. Dr. Wernicke attempted such a proof on the principle that the four classifications involved (viz. into rows, columns, officers' rank and regiments) are tactically equivalent. Hence applications become possible to more complex problems involving an interference of classifications.

2. Dr. Wernicke defines a "three-dimensional map" to be a division of three-dimensional space by any number of surfaces meeting in lines only. The latter meet in points only, which we call the vertices of the map. The portions of the lines extending from any vertex to the consecutive one are the edges; the portions of the surfaces bounded by the edges, the partitions; the portions of space partitioned off, the cells or districts of the map. If a map contains neither multiply-connected cells nor partitions, its cells may be filled with nine different gases in such a way

that no partition separates cells holding the same gas. The method of proof is similar to that of the corresponding four-color theorem of plane maps.

3. In this paper Professor Osgood gives an account of the existing proofs of the formulas for differentiating a definite integral with respect to a parameter, followed by a critique of these proofs. He then gives a new proof based on Green's theorem, which is free from some of the objections which may be brought against the older proofs. The paper will be offered to the *Annals of Mathematics*.

4. Charles Hutton attributed the invention of the straight-edge slide rule to Edmund Wingate, but gave no references to Wingate's publications. Augustus De Morgan attributed the invention to William Oughtred and denied that Wingate ever wrote on the slide rule. Professor Cajori was able to quote from one of Wingate's works of 1628 and to show that Wingate was the first inventor. The "runner," often attributed to Mannheim (1850), was shown to be the invention of I. Newton and E. Stone. Probably the first inversion of a logarithmic line is found in Everard's slide rule, used in England in gauging during the latter part of the seventeenth and in the eighteenth century. Newton's and Stone's adaption of the slide rule to the solution of numerical equations was explained. Finally, the time of the introduction of the slide rule into the United States was considered.

5. Dr. Ingold considers the ratio of areas of triangles formed from six points, A, B, C, P, Q, R, in a plane. The double ratio $[PA, BC, QR]$ is defined to be

$$\frac{PBC}{ABC} \div \frac{PQR}{AQR},$$

which can be proved to be invariant under projection. The possible double ratios are seen to be equal in sets of eight; there remain therefore ninety which are distinct. From identities such as $ABC.PQR = PBC.AQR + QBC.PAR + RBC.PQA$ a number of relations are easily established among the different double ratios, so that not more than four of the ninety are independent, the others being expressible in terms of these.

6. In any system of spherical coordinates the aberrational corrections may be computed from the corresponding formulas for the parallaxes by differentiating these formulas with respect to the time, treating the angular coordinates as constant. Dr. James obtains the following results:

$$\Delta A_a = \frac{d}{dt}\left(\frac{rP}{\cos B - Q\sin B + R\cos B}\right)$$

$$V\Delta B_a = \frac{drQ}{dt} - \frac{drP}{dt}\sin B \tan\frac{\Delta A_a}{2}.$$

7. Any quaternion $q = w + xi + yj + zk$ may be written in the form $q = w + xi + (y + zi)j$. We may represent a quaternion graphically therefore by representing $w + xi$ and $y + zi$ by the ordinary method of representing complex numbers, $w + xi$ being drawn on one plane and $y + zi$ on a second plane, called the j-plane. For convenience we may divide an ordinary plane into four quadrants, the first quadrant being the plane of $w + xi$, the third the quadrant of $y + zi$. The second and fourth quadrants, by orthogonal projections, furnish also representations for $x + yk$, and $w + zk$. The product and the sum of two quaternions may be found graphically by methods similar to those in use in the theory of complex numbers. The theory of two independent complex variables finds a graphical representation here also. Finally the descriptive geometry of four dimensional space and the quaternion algebra as applied to four-dimensional space come into intimate contact, and analytic processes may be easily interpreted in geometric procedures, or conversely. The paper will be offered to the *Transactions*.

8. In this paper, Professor Hedrick offers a modification of the definition of the jacobian suggested by Jordan (Cours d'analyse volume 1, § 148) and by Porter (*L'Enseignement mathématique*, volume 9, part 4, page 272). The jacobian is defined as the limit of the quotient of two areas, one of which is a right triangle in the original plane. This limit exists whenever the ordinary jacobian exists, and á fortiori whenever the limit used by Jordan and Porter exists. Certain advantages are pointed out.

9. Professor Chessin's paper appears in full elsewhere in the present number of the BULLETIN.

10. By a theorem of Professor Van Vleck (*Annals of Mathematics*, July, 1903), an algebraic equation with real coefficients $c_0 + c_1x + c_2x^2 + \cdots + c_nx^n = 0$ will have all or all but one of its roots (according as n is even or odd) imaginary if all the members of a certain set of determinants are positive. Professor Kellogg deduces from Descartes's rule of signs a necessary condition that all the roots of such an equation be real; namely that all the determinants $c_{j-1}c_{j+1} - c_j^2 < 0$ $(k < j < n$, where c_k is the first coefficient different from 0). A second, sharper condition, of which the first is a consequence, is that none of the quadratics

$$\begin{vmatrix} x^2 & x & 1 \\ c_{j-1} & c_j & c_{j+1} \\ c_{j+1} & c_{j+1} & c_{j+2} \end{vmatrix} = 0 \qquad (k \leqq j < n, \ c_{-1} = c_{n+1} = 0)$$

has real roots. The paper will appear in the January number of the *Annals of Mathematics*.

11. In Dr. Ingold's second paper a quadratic differential form D with constant coefficients is considered. The differential parameters of such a form which involve no functons except algebraic forms $b, c, \cdots$ are concomitants of these forms and of the quadratic algebraic form a having the same coefficients as D. Conversely, any concomitant of the system $a, b, c, \cdots$, when multiplied by a power of the discriminant of D, becomes a differential parameter of D. Relations among the differential parameters of D, when thus specialized, yield relations among the concomitants of the system $a, b, c, \cdots$.

12. Professor Newson's paper seeks to determine invariants of collineations. As one such, is found a certain cross ratio, and it is shown that this cross ratio for the product of two collineations is a simple function of the two corresponding cross ratios for the two collineations determining the product.

13. The paper by Professor Miller is a continuation of his article entitled "On the holomorph of a cyclic group" which was published in the *Transactions*, volume 4 (1903), page 151. While the holomorph of a cyclic group plays a fundamental role in many group theory considerations, it also involves all questions in regard to the exponent to which numbers belong with respect to a given arbitrary modulus m. For instance,

the numbers which Epstein, in his recent article published in the *Archiv der Mathematik und Physik*, calls primitive roots of m are those which correspond to the operators of highest order in the group of isomorphisms of the cyclic group of order m, and hence the determination of the number of such primitive roots is a very special case of the determination of the number of operators of a given order in an abelian group. In the present paper special attention is paid to the group of isomorphisms of the holomorph of the cyclic group of order 2^m, and one of the most important results is stated as follows: The group of isomorphisms of the holomorph of the cyclic group of order 2^m is the direct product of the group of order 2 and the group of cogredient isomorphisms of the double holomorph of the cyclic group of order 2^m. This paper will appear in the *Transactions*.

14. In this paper Professor Davis studies the connection between his theory of colored imaginaries and the pole and polar theory of the cubic curve. If $f(x, y, z) = 0$ is the equation of the cubic, $\Delta'(x, y, z)$ the first polar of x', y', z' with regard to $f(x, y, z)$, while Δ'' is the polar of x'', y'', z'', then when we write $x = x' + ix''$, etc., we get $f(x', y', z') = \Delta'(x'', y'', z'')$ and $f(x'', y'', z'') = \Delta''(x', y', z')$. It is upon the basis of these two equations that the paper is built.

O. D. KELLOGG,
Secretary of the Section.

NOTE ON THE COMPOSITION OF FINITE ROTATIONS ABOUT PARALLEL AXES.

BY PROFESSOR ALEXANDER ZIWET.

1. IT is well known that the succession of two finite rotations of a rigid plane figure in its plane (or, what amounts to the same, of a rigid body about parallel axes), say a rotation of angle θ' about a point O' followed by a rotation θ'' about O'', is equivalent to a single rotation of angle $\theta = \theta' + \theta''$ about a point O. The center O is found as the intersection of the lines obtained by turning $O'O''$ about O' through an angle $-\frac{1}{2}\theta'$ and $O''O'$ about O'' through $+\frac{1}{2}\theta''$.

As a clockwise rotation of angle ϕ is equivalent to a counterclockwise rotation of angle $2\pi - \phi$, the angles of rotation can all be made of the same sense, say counterclockwise; and as a rotation of angle $\phi + 2\kappa\pi$ is equivalent to the rotation of angle ϕ, the angles can be confined to values between 0 and 2π.

The construction of the point O is therefore always possible unless $\theta = \theta' + \theta''$ is $\equiv 0 \pmod{2\pi}$. In this limiting case the center lies at infinity, and the resultant displacement is a translation whose vector is readily determined.

2. When there are more than two successive rotations, the resultant of the first and second rotations can be compounded by the same method with the third rotation, and so on. It is apparent that the angle of the final resultant rotation is the sum of the angles of the given rotations, but the construction of the center becomes rather complicated. It is the object of the present note to indicate a convenient method for finding this center, by applying vector methods systematically.

3. A single rotation, θ' about O' (Fig. 1), carries any point A of the rigid figure from the initial position A_0 to the final

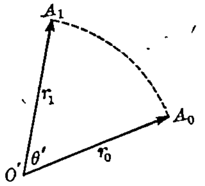

FIG. 1.

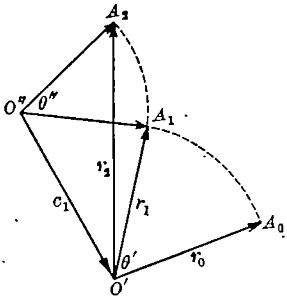

FIG. 2.

position A_1. Taking the fixed point O' as origin and putting $A_0 - O' = \mathbf{r}_0$, $A_1 - O' = \mathbf{r}_1$, we have the vector equation

$$\mathbf{r}_1 = e^{i\theta'}\mathbf{r}_0,$$

which means that the operator $e^{i\theta'}$ applied to the vector $\mathbf{r}_0$ turns it in the plane through the angle θ'.*) This is of course merely another form of stating the ordinary method of complex numbers.

Two successive rotations, θ' about O' and θ'' about O'' (Fig. 2), carry the point A from A_0 to A_1 and from A_1 to A_2. With $O' - O'' = \mathbf{c}_1$, $A_2 - O' = \mathbf{r}_2$ we have $A_1 - O'' = \mathbf{r}_1 + \mathbf{c}_1$,

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[*] G. Peano, Gli elementi di calcolo geometrico, Torino, 1891, p. 19.

$A_2 - O'' = \mathbf{r}_2 + \mathbf{c}_1$, whence

$$\mathbf{r}_2 + \mathbf{c}_1 = e^{i\theta''}(\mathbf{r}_1 + \mathbf{c}_1),$$

or, replacing $\mathbf{r}_1$ by its value from the preceding equation,

$$\mathbf{r}_2 = e^{i(\theta''+\theta')}\mathbf{r}_0 + e^{i\theta''}\mathbf{c}_1 - \mathbf{c}_1.$$

In the case of three successive rotations, θ' about O', θ'' about O'', θ''' about O''', putting $O'' - O''' = \mathbf{c}_2$, $A_3 - O' = \mathbf{r}_3$ we have

$$\mathbf{r}_3 + \mathbf{c}_1 + \mathbf{c}_2 = e^{i\theta'''}(\mathbf{r}_2 + \mathbf{c}_1 + \mathbf{c}_2),$$

whence eliminating $\mathbf{r}_2$ we find

$$\mathbf{r}_3 = e^{i(\theta'''+\theta''+\theta')}\mathbf{r}_0 + e^{i(\theta'''+\theta'')}\mathbf{c}_1 + e^{i\theta'''}\mathbf{c}_2 + \mathbf{c}_3,$$

where $\mathbf{c}_3 = -\mathbf{c}_1 - \mathbf{c}_2 = O''' - O'$.

Similarly we find for four successive rotations, θ' about O', $\cdots$, θ^{iv} about O^{iv}

$$\mathbf{r}_4 = e^{i(\theta^{\text{iv}}+\theta'''+\theta''+\theta')}\mathbf{r}_0 + e^{i(\theta^{\text{iv}}+\theta'''+\theta'')}\mathbf{c}_1 + e^{i(\theta^{\text{iv}}+\theta''')}\mathbf{c}_2 + \mathbf{c}^{i\theta^{\text{iv}}}\mathbf{c}_3, + \mathbf{c}_4,$$

where $\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 + \mathbf{c}_4 = 0$. The vectors $\mathbf{c}_1, \cdots, \mathbf{c}_4$ are equal to the sides of the quadrilateral formed by the centers $O', \cdots, O^{\text{iv}}$; these centers are points of the fixed plane, not of the moving figure. The initial and final radii vectores $\mathbf{r}_0$, $\mathbf{r}_4$ of A are drawn from O'. If any other point Q of the plane were taken as origin, it would only be necessary to replace $\mathbf{r}_0$, $\mathbf{r}_4$ by $\mathbf{r}_0' - \mathbf{q}$, $\mathbf{r}_4' - \mathbf{q}$, where $\mathbf{r}_0'$, $\mathbf{r}_4'$ are the radii vectores of A drawn from Q, and $\mathbf{q} = O' - Q$.

4. In the expression found for $\mathbf{r}_4$, all terms after the first are independent of the particular point A of the figure. Their sum represents a vector $\mathbf{s}_4$ which can be written

$$\mathbf{s}_4 = e^{i\theta^{\text{iv}}}(e^{i\theta'''}(e^{i\theta''}\mathbf{c}_1 + \mathbf{c}_2) + \mathbf{c}_3) + \mathbf{c}_4.$$

This form indicates the most convenient way of constructing the vector $\mathbf{s}_4$ (Fig. 3): turn $O' - O'' = \mathbf{c}_1$ about O'' through θ'' and add $\mathbf{c}_2$; then turn the sum so obtained about O''' through θ''' and add $\mathbf{c}_3$; finally turn the vector so obtained about O^{iv} through θ^{iv} and add $\mathbf{c}_4$; this gives the vector $\mathbf{s}_4 = O_4 - O'$ which evidently represents the displacement of that point of the rigid figure which originally coincided with O'.

Denoting the angle of the resultant rotation by θ, so that $\theta \equiv \theta' + \theta'' + \theta''' + \theta^{\text{iv}} (\text{mod } 2\pi)$ and introducing the vector $\mathbf{s}_4$, we have the simple result

$$\mathbf{r}_4 = e^{i\theta}\mathbf{r}_0 + \mathbf{s}_4.$$

5. It is obvious that in the case of n successive rotations we have similarly

$$\mathbf{r}_n = e^{i\theta}\mathbf{r}_0 + \mathbf{s}_n,$$

where $\theta \equiv \theta' + \theta'' + \cdots + \theta^{(n)}$ and

$$\mathbf{s}_n = e^{i\theta^{(n)}}\{\cdots [e^{i\theta'''}(e^{i\theta''}\mathbf{c}_1 + \mathbf{c}_2) + \mathbf{c}_3] + \cdots + \mathbf{c}_{n-1}\} + \mathbf{c}_n.$$

Thus, the final radius vector of any point of the rigid figure is found by turning its initial radius vector about O' through an angle $\theta \equiv \theta' + \theta'' + \cdots + \theta^{(n)}$ and adding to it the vector $\mathbf{s}_n$. In other words, the resultant displacement is resolved into the rotation of angle θ about O' and the translation $\mathbf{s}_n$ of O'.

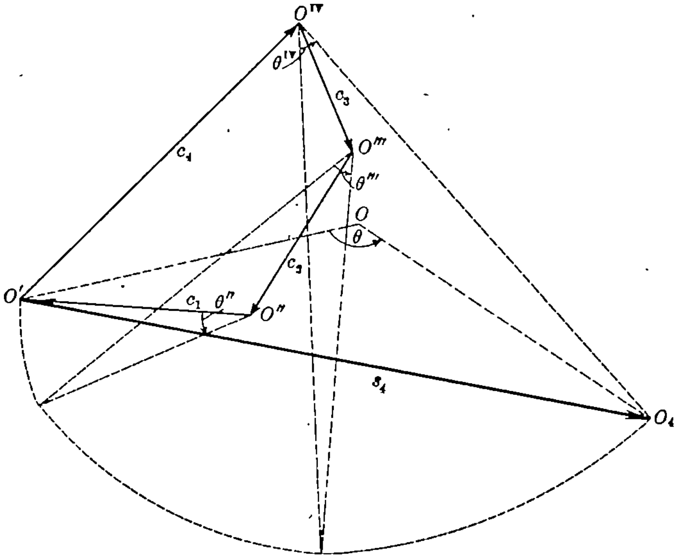

FIG. 3.

The center O of the equivalent single rotation being a fixed point its radius vector $\mathbf{r}$ is found by putting $\mathbf{r}_n = \mathbf{r}_0 = \mathbf{r}$ which giv

$$(1 - e^{i\theta})\mathrm{r} = \mathrm{s}_n, \quad \text{or} \quad \mathrm{r} = \frac{1}{2\sin\frac{1}{2}\theta}\, e^{i(\pi-\theta)/2}\mathrm{s}_n.$$

This means that the point O is the vertex of the isosceles triangle whose base is $O'O_n$ and whose angle at O is θ if $\theta < \pi$ and is $2\pi - \theta$ if $\theta > \pi$. In the former case the sense $O'O_nO$ is positive, in the latter it is negative. If $\theta = \pi$, the resultant displacement is a reversal (Umwendung) about the midpoint of $O'O_n$. If $\theta \equiv 0 \pmod{2\pi}$ the resultant displacement reduces to the translation s_n.

If, in particular, the angles of rotation θ'', θ''', $\cdots$, $\theta^{(n)}$ are equal, respectively, to the exterior angles of the polygon of centers at O'', O''', $\cdots$, $O^{(n)}$, the vector s_n has, as appears at once from its construction (Fig. 3), the direction and sense of c_n and a length equal to the perimeter of the polygon $O'\,O'' \cdots O^{(n)}$. If, in addition, θ' is equal to the exterior angle at O' and the polygon is convex, the resultant rotation is zero, and the resultant displacement reduces to the translation s_n.

ANN ARBOR, *December*, 1907.

ON AN INTEGRAL APPEARING IN PHOTOMETRY.

BY PROFESSOR A. S. CHESSIN.

(Read before the Southwestern Section of the American Mathematical Society, November 30, 1907.)

IN the course of research on the variation of the intensity of illumination, Mr. E. P. Hyde found it necessary to evaluate the integral

$$\int \frac{i \cos\phi \cos\theta}{r^2}\, dS$$

taken over a part of the surface of a cylinder.† Having encountered some difficulties on the mathematical side of the problem, Mr. Hyde requested the author to compute the value of this integral. As this computation contains some features of interest to mathematicians the solution is here presented without, however, referring to the physical aspects of the problem. The purely mathematical problem, then, may be stated as follows:

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† "Talbot's law as applied to the rotating sectored disc," by E. P. Hyde, *Bulletin of the U. S. Bureau of Standards*, vol. 2, No. 1.

Evaluate the integral

$$J = ia\int_{-h}^{h} dy \int_{-\cos^{-1} a/l}^{\cos^{-1} a/l} \frac{(a - l\cos\alpha)(l - a\cos\alpha)d\alpha}{(a^2 + l^2 - 2al\cos\alpha + y^2)^2}.$$

By means of the substitutions

$$\tan\tfrac{1}{2}\alpha = x, \quad \frac{(l+a)^2 + y^2}{(l-a)^2 + y^2} = z,$$

$$\frac{l+a}{l-a} = p, \quad \frac{(l+a)^2 + h^2}{(l-a)^2 + h^2} = q,$$

the given integral is transformed into

$$i\left(1 - \frac{a}{l}\right)\int_{p^2}^{q} \sqrt{\frac{z-1}{p^2 - z}}\, dz \int_0^{1/\sqrt{p}} \frac{(p^2x^4 - 1)dx}{(x^2+1)(zx^2+1)^2}$$

and, after effecting the integration with regard to x, becomes

$$i(1 - a/l)(J_1 + J_2 + J_3),$$

where

$$J_1 = (p^2 - 1)\tan^{-1}\frac{1}{\sqrt{p}}\int_{p^2}^{q} \frac{dz}{(z-1)\sqrt{(z-1)(p^2-z)}},$$

$$J_2 = \tfrac{1}{2}\sqrt{p}\int_{p^2}^{q} \frac{(p-z)dz}{z\sqrt{(z-1)(p^2-z)}},$$

$$J_3 = \int_{p^2}^{q}\left[\frac{z-p^2}{z-1} - \frac{p^2+z^2}{2z}\right]\frac{\tan^{-1}\sqrt{z/p}\,dz}{\sqrt{z(z-1)(p^2-z)}}.$$

The integrals J_1 and J_2 are readily evaluated by the ordinary rules. The third integral J_3 apparently offers some difficulties. The integration, however, is very simple and does not even require the introduction of elliptic functions.

In fact, consider the integral

$$U = \int\left[\frac{z-p^2}{z-1} - \frac{p^2+z^2}{2z}\right]\frac{dz}{\sqrt{z(z-1)(p^2-z)}}.$$

It belongs to a class of pseudo-elliptic integrals discussed by

Euler,[*] Raffy,[†] and Goursat.[‡] Adopting Goursat's method, we have in this case three relations of involution by which the roots of the equation $z(z-1)(p^2-z)=0$ are permuted in pairs. Two of these relations are

$$z'z''=p^2, \quad z'z''-(z'+z'')+p^2=0.$$

On the other hand, if we put

$$\frac{z-p^2}{z-1}-\frac{p^2+z^2}{2z}=f_1(z)+f_2(z),$$

where

$$f_1(z)=\frac{z-p^2}{z-1}-z, \quad f_2(z)=\tfrac{1}{2}\left(z-\frac{p^2}{z}\right),$$

we note that each one of the functions f_1 and f_2 satisfies a relation, corresponding to one or the other of the two relations of involution given above, viz.,

$$f_1(z)+f_1\left(\frac{p^2-z}{1-z}\right)\equiv 0, \quad f_2(z)+f_2\left(\frac{p^2}{z}\right)\equiv 0.$$

Hence,[§] the integral U is pseudo-elliptic.

At the same time, if we apply the author's reduction formulas given in the pages of this BULLETIN,[||] viz., formulas (5) and (6), where we should put $m=r=3$, $\Sigma n_k=s=\rho=2$, and, consequently, $q=1$, we find that

$$U=\frac{(Az+B)\sqrt{z(z-1)(p^2-z)}}{z(z-1)}$$

$$+\lambda_0\int\frac{dz}{\sqrt{z(z-1)(p^2-z)}}+\lambda_1\int\frac{zdz}{\sqrt{z(z-1)(p^2-z)}}.$$

But the two integrals on the right-hand side are obviously elliptic. Hence, since it has been shown that U is pseudo-elliptic, we must have $\lambda_0 = \lambda_1 = 0$, so that, barring a constant of integration,

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[*] "Sur une formule d'Euler," par M. Hermite, *Journal de Liouville*, 1880.

[†] "Sur les transformations invariantes des différentielles elliptiques," par L. Raffy, *Bulletin de la Soc. Math. de France*, 1884.

[‡] "Sur les integrales pseudo-elliptiques," par E. Goursat, *Bulletin de la Soc. Math. de France*, 1885. See also Goursat, Cours d'analyse, vol. 1, pp. 264–266.

[§] L. c.

[||] "Note on hyperelliptic integrals," BULLETIN, vol. 4, pp. 93–96.

$$U = (Az + B)\sqrt{\frac{p^2 - z}{z(z-1)}}.$$

It now only remains to determine the coefficients A and B. This is readily done by comparing both sides of the identity

$$\frac{z - p^2}{z - 1} - \frac{p^2 + z^2}{2z} = \sqrt{z(z-1)(p^2 - z)}\frac{d}{dz}\left[(Az + B)\sqrt{\frac{p^2 - z}{z(z-1)}}\right]$$

which gives $A = B = 1$ and, therefore, barring the constant of integration,

$$U = (z + 1)\sqrt{\frac{p^2 - z}{z(z-1)}}.$$

Returning now to the integration of J_3 we observe that

$$J_3 = \int_{p^2}^{q} \tan^{-1}\sqrt{\frac{z}{p}}\, dU = \left[U \tan^{-1}\sqrt{\frac{z}{p}}\right]_{p^2}^{q} - \tfrac{1}{2}\sqrt{p}\int_{p^2}^{q}\frac{U dz}{(z+p)\sqrt{z}}$$

$$= (q+1)\sqrt{\frac{p^2 - q}{q(q-1)}}\tan^{-1}\sqrt{\frac{q}{p}} + \tfrac{1}{2}\sqrt{p}\int_{p^2}^{q}\frac{(z+1)(z-p^2)dz}{z(z+p)\sqrt{(z-1)(p^2-z)}}$$

or, adding the integral J_2,

$$J_2 + J_3 = (q + 1)\sqrt{\frac{p^2 - q}{q(q-1)}}\tan^{-1}\sqrt{\frac{q}{p}}$$

$$- \tfrac{1}{2}(p^2 - 1)\sqrt{p}\int_{p^2}^{q}\frac{dz}{(z + p)\sqrt{(z-1)(p^2 - z)}}.$$

It is a simple matter now to complete the integration. The result is *

$$J = \frac{i}{l}\left\{2a \tan^{-1}\frac{h}{\sqrt{l^2 - a^2}} + h\left(\sqrt{q} + \frac{1}{\sqrt{q}}\right)\tan^{-1}\sqrt{\frac{q}{p}} - 2h \cot^{-1}\sqrt{p}\right\}.$$

SAINT LOUIS,
November 30, 1907.

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* It is given in Mr. Hyde's paper in a slightly different but equivalent form. L. c., p. 8.

HERMITIAN FORMS WITH ZERO DETERMINANT.

BY PROFESSOR J. I. HUTCHINSON.

(Read before the American Mathematical Society, December 28, 1907.)

VERY little has yet been attempted in the arithmetic determination of infinite discontinuous groups on two or more variables. The earliest memoir on this subject, by Picard,* treats of groups which leave certain ternary Hermitian forms invariant. Such forms can always be reduced, as Picard has remarked in a later paper,† to the form

$$(1) \qquad \alpha u\bar{u} + \beta v\bar{v} - \gamma w\bar{w}, \qquad (\alpha, \beta, \gamma > 0),$$

if the determinant of the form is negative. The locus of singular (or limit) points for the group is the spread obtained by equating (1) to zero. It is the object of the present note to determine the kind of groups which have their limit points on such a quadric spread in case the corresponding Hermitian form has a zero determinant.

Let the given form f be represented by

$$(2) \qquad f \equiv \sum_{i,\,k=1}^{n} f_{ik} x_i \bar{x}_k,$$

in which $\bar{x}$ is the conjugate of x, and the coefficient f_{ik} is the conjugate of f_{ki}. The determinant $D = |f_{ik}|$ of this form we assume to be zero and of rank k. Accordingly the elements of a certain column in D (the gth, say) are a linear combination of the corresponding elements of the other columns, that is,

$$f_{ig} = \sum_{\lambda} \bar{a}_\lambda f_{i\lambda} \qquad \left(\begin{smallmatrix}\lambda = 1,\, 2,\, \ldots,\, g-1,\, g+1,\, \ldots,\, n\\ i = 1,\, 2,\, \ldots,\, n\end{smallmatrix}\right),$$

in which the $\bar{a}_\lambda$ are finite and g is a suitably chosen integer.

If now we make the substitution

$$(3) \qquad \begin{aligned} \eta_\lambda &= x_\lambda + a_\lambda x_g \quad (\lambda = 1, 2, \cdots, g-1, g+1, \cdots, n), \\ \eta_g &= x_g, \end{aligned}$$

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* "Sur une classe de groupes discontinues de substitutions linéaires, etc.," *Acta Mathematica*, vol. 1 (1881), p. 297.

† "Sur les formes quadratiques ternaires indéfinies à indéterminées conjuguées, etc.," *Acta Mathematica*, vol. 5 (1885), p. 121.

the form f reduces to

$$(4) \qquad \sum_{\lambda,\,\mu} f_{\lambda\mu}\eta_\lambda\bar{\eta}_\mu \quad (\lambda,\ \mu = 1,\ 2,\ \cdots,\ g-1,\ g+1,\ \cdots,\ n),$$

as may readily be verified by substituting in (4) the preceding expressions for η_λ and using the relations

$$\sum_\lambda f_{\lambda\mu} a_\lambda = \sum_\lambda \bar{f}_{\mu\lambda} a_\lambda = \bar{f}_{\mu g},$$

$$\sum_{\lambda\mu} f_{\lambda\mu} a_\lambda \bar{a}_\mu = \sum_\lambda a_\lambda f_{\lambda g} = \bar{f}_{gg} = f_{gg}.$$

The determinant of (4) is the first minor F_{gg} corresponding to f_{gg} in D. If the rank k of D is less than $n-1$, F_{gg} is zero and the process is continued until we reach a form whose determinant $M_{\mu\mu}$ does not vanish. This determinant is evidently a diagonal minor of D of order k, and accordingly its elements are common to the rows and columns numbered $\mu_1, \mu_2, \cdots, \mu_k$, these k integers being selected from the set $1, 2, \cdots, n$.

It is clear that by choosing different values for g in the transformation (3), and in the subsequent ones, the reduction could be made in a variety of ways and the determinant of the final form may be any non-vanishing diagonal minor of D of order k. We have now to show that, *in whatever way the given Hermitian form is reduced, the determinant of the final form always has the same sign.* This follows from a well-known theorem in determinants. For, let $M_{\nu\nu}$ be the diagonal minor of order k whose elements are taken from the rows and columns numbered $\nu_1, \nu_2, \cdots, \nu_k$. Denote by $M_{\mu\nu}$ the minor

$$\begin{vmatrix} a_{\mu_1\nu_1} & a_{\mu_1\nu_2} & \cdots & a_{\mu_1\nu_k} \\ a_{\mu_2\nu_1} & a_{\mu_2\nu_2} & \cdots & a_{\mu_2\nu_k} \\ \cdot & \cdot & \cdot & \cdot \\ a_{\mu_k\nu_1} & \cdots & & a_{\mu_k\nu_k} \end{vmatrix}$$

and by $M_{\nu\mu}$ the minor obtained from this by interchanging μ_i and ν_i $(i = 1, 2, \cdots, k)$. By interchanging rows and colums in $M_{\nu\mu}$, it is readily seen to be the numerical equivalent of $\bar{M}_{\mu\nu}$.

But since the determinant of the four minors thus obtained is zero,* we have

$$(5) \qquad M_{\mu\mu}M_{\nu\nu} = M_{\mu\nu}\bar{M}_{\mu\nu}.$$

The right member is positive or zero and hence if $M_{\mu\mu}$ and $M_{\nu\nu}$ be not zero they must have the same sign.

Suppose now that (2) has been reduced to the form

$$(6) \qquad \sum_{i,j=n-k+1}^{n} f_{ij}y_iy_j,$$

the determinant of which $|f_{ij}|$ is not zero. A transformation on the x's which leaves the form (1) unaltered will be replaced by a corresponding transformation on the y's of the form

$$(7) \qquad \begin{aligned} y'_\lambda &= \sum_{\mu=\lambda}^{n} a_{\lambda\mu}y_\mu \qquad (\lambda = 1, 2, \cdots, n-k), \\ y'_i &= \sum_{j=n-k+1}^{n} a_{ij}y_j \quad (i = n-k+1, \cdots, n). \end{aligned}$$

A closer characterization of the group can be made sufficiently clear by an examination of the simplest case, $n = 3$. The general substitution of the group may be represented by

$$(8) \quad x' = ax + by + cz, \quad y' = ay + \beta z, \quad z' = \gamma y + \delta z.$$

Consider first the substitutions of the form

$$(9) \qquad x' = a'x + b'y + c'z, \quad y' = y, \; z' = z,$$

the totality of which form a subgroup of (8). As a group ot transformations on the single variable x, it must be contained in the cyclic or the parabolic rotation groups.† It follows then that either b', c' are zero, or that a can have only the values ± 1 and the remaining terms in x' are expressible in the form

$$b'y + c'z = m\omega + n\omega'$$

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* E. Pascal, Die Determinanten, Leipzig, 1900, p. 195. The determinant D might be called skew conjugate, and includes the symmetric and the skew-symmetric (after multiplying all the elements by i) as special cases. The theorem formulated in (5) would specialize in these cases as follows: *If a symmetric, or a skew-symmetric, determinant of order n is of rank $k < n$, then all diagonal minors of order k which do not vanish have the same sign.*

† Fricke-Klein, Automorphe Functionen, I, p. 214 ff.

in which ω, ω' are arbitrary, fixed parameters and m, n are integers. Hence, *on replacing* y, z *by suitable variables* ω, ω', *the group* (8) *reduces to one of the forms*

$$(10) \qquad \begin{vmatrix} a & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \delta & \gamma \end{vmatrix}, \qquad \begin{vmatrix} \pm 1 & m & n \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{vmatrix}.$$

If the group is of the first type, it is completely reducible * and of no interest as a ternary group. If of the second type, the coefficients $\alpha, \beta, \gamma, \delta$ must be integers such that $\alpha\delta - \beta\gamma = 1$, otherwise the combination of two substitutions would give one whose determinant does not have integers in its first row, and the group would contain infinitesimal transformations.†

The method of reasoning which we have applied to ternary groups is readily extended to the case of n variables and gives the result:

Either the group is completely reducible and has the form ‡

$$\begin{vmatrix} G & 0 \\ 0 & G' \end{vmatrix}$$

in which G' *is the group of the reduced form* (6) *and* G *is any discontinuous group in the remaining* $n - k$ *variables, or it has the form*

$$\begin{vmatrix} G_{11} & 0 & 0 \\ 0 & G_{22} & G_{23} \\ 0 & 0 & G_{33} \end{vmatrix},$$

where G_{11} *is any discontinuous group on* p *of the variables* $(p < n - 2)$, *and the other matrices have the form*

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* See W. Burnside, *Acta Mathematica*, vol. 28 (1904), p. 369, and A. Loewy, *Transactions*, vol. 6 (1905), p. 505.

† The group thus determined is of particular interest on account of its connection with the theory of elliptic functions. In fact, the Weierstrass functions constitute a complete system of invariants for the group. See Klein-Fricke, Modulfunctionen, vol. 2, p. 3 ff.

‡ For this very convenient symbolic representation see A. Loewy, *Transactions*, vol. 4 (1903), p. 44.

$$G_{22}=\begin{vmatrix}\pm 1 & 0 & 0 & \cdots\\ 0 & \pm 1 & 0 & \cdots\\ \cdot & \cdot & \cdot & \cdot\\ 0 & \cdot\;\cdot\;\cdot & & \pm 1\end{vmatrix},\quad G_{23}=\begin{vmatrix}0 & 0 & \cdots & m_1 & n_1\\ 0 & 0 & \cdots & m_2 & n_2\\ & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & \cdots & m_r & n_r\end{vmatrix},$$

$$G_{33}=\begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix},$$

in which $m_i\, n_i\; \alpha,\ \beta,\ \gamma,\ \delta$ *are integers and* $\alpha\delta-\beta\gamma=1$.

CORNELL UNIVERSITY,
December, 1907.

TWO TETRAEDRON THEOREMS.

BY PROFESSOR HENRY S. WHITE.

THE sphere and the tetraedron yield two combinations familiar to students of geometry, those in which one object is inscribed in the other; and one less well known, that in which the edges of the tetraedron are tangents to the sphere. A novel theorem upon the circumscribed tetraedron was propounded in 1897 by Bang and proved by Gehrke, and has been made the starting-point for extended developments by Franz Meyer* (1903) and Neuberg (1907).† It is this: If the contact point in each face of a tetraedron circumscribed about a sphere be joined by a straight line to each vertex in its face, then three angles at each contact point are equal respectively to the three formed at any other contact point. Or it may be stated thus: Opposite edges of a circumscribed tetraedron subtend equal angles at the points of contact of the faces which contain them.

While elementary proofs of this are interesting, a more elaborate deduction is of value here as suggesting a second theorem. It can be made to depend upon the well-known theorem from the projective geometry of a straight line, namely,

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[*] W. Franz Meyer: "Ueber Verallgemeinerungen von Sätzen über die Kugel und das ein- resp. umbeschriebene Tetraeder." *Jahresbericht der deutschen Mathematiker-Vereinigung*, 1903, p. 137.

[†] J. Neuberg: "Ueber die Berührungskugeln eines Tetraeders," *Jahresbericht der deutschen Mathematiker-Vereinigung*, 1907, p. 345.

that an involutory projective relation can be uniquely determined by two arbitrary pairs of corresponding points.

All points, real and imaginary, of a line are represented in a one-to-one way by the real points of a plane; and thence, through stereographic projection (or inversion) by the real points of a sphere. Linear or projective transformations of points on the line are equivalent to those real linear transformations in space of three dimensions which carry the sphere over into itself, and vice versa. *These transformations do not alter the magnitude of any angle between curves on the sphere.* To apply these facts toward proof of the theorem under consideration, let the circumscribed tetraedron have vertices A, B, C, D, and for points of contact, A_1 in the face BCD, B_1 in ACD, etc.

We wish to show, for example, that three angles at A_1 made by rays A_1B, A_1C, and A_1D are equal respectively to three angles formed at B_1 by rays drawn to A, D, and C. All these rays are of course tangent to the sphere.

Determine an involution by the pairs of points A_1 and B_1, C_1 and D_1 It will carry planes into planes, and planes tangent to the sphere into other such planes; in particular the four faces of the tetraedron are permuted among themselves, so that the vertices A and B are exchanged, likewise C and D. Hence the angles on the spherical surface measured by BA_1C, CA_1D, DA_1B are transformed into angles of unchanged magnitude measured respectively by AB_1D, DB_1C, CB_1A, and this proves the theorem.

This same method may be applied to draw new relations from the theorem that a collineation can be found which shall transform three given points of a straight line into three arbitrary points. Upon the sphere let the six edges of a tetraedron be tangent in six points, which may be denoted most clearly by double indices, 12, 13, 14, 34, 42, 23; *e. g.*, the edge AB shall touch the sphere at 12, etc. Through each point of contact pass a plane containing the opposite edge. We shall prove that *these planes form three pairs which intersect the sphere in orthogonal pairs of circles; and that the right angles between these circles at any contact are bisected by the traces of two other planes, each containing four points of contact.*

The relations appear on inspection as true for the regular tetraedron; and to this form every set of tangent edges of an oblique tetraedron can be brought by the admissible transformations. For the three contacts 12, 23, 31 can be brought by such a transformation into coincidence with the contacts of three base edges of a regular tetraedron. These three points determine then the vertex 4, and with it the other three contacts; not uniquely, but as one of two definite positions: an interchange of two contacts, if necessary, will bring the vertex 4 to coincidence with that of the regular tetraedron. By this collinear transformation all the planes above mentioned come to the positions of the nine correspondingly situated planes in the regular figure, thus proving the theorem.

It is to be noticed that this tetraedron whose edges all touch a sphere is not an arbitrary figure of its class, since it depends on six constants, beside the radius of the sphere and three for the center, — ten in all, while the tetraedron with four vertices free involves twelve constants. Two sufficient conditions for inscriptibility of a sphere in this sense are these, that the sum of the lengths of two opposite edges shall be the same for each of the three opposite pairs. Less symmetrically, we note that circles inscribed in any two lateral faces must be tangent to that inscribed in the base, and from this will follow the same relation in the third lateral face.

The set of six contacts of six edges of a tetrahedron with a sphere divides naturally into threes in four ways; for example, 12, 13, 14; 23, 34, 42, three in one face and three on edges having the opposite vertex in common. *Each set of three may exchange rôles with its complementary set of three*, as is seen by applying to the figure the polar reciprocal transformation with respect to the sphere, rotating each edge through a right angle to the conjugate position. The three planes through four points of contact go into poles, whose relations to the six contacts might invite attention.

Finally it may be recalled that all the constructions suggested by these figures can be carried out in the plane, upon the six contacts of four mutually tangent circles; also that the complementary sets of three points indicate an interesting problem in the covariants of a binary cubic form.

VASSAR COLLEGE,
December, 1907.

SINGULAR POINTS OF A SIMPLE KIND OF DIFFERENTIAL EQUATION OF THE SECOND ORDER.

BY PROFESSOR C. A. NOBLE.

(Read before the San Francisco Section of the American Mathematical Society, September 28, 1907.)

IN a series of four memoirs in the *Journal de Mathématiques* (series 3, numbers 7, 8; series 4, numbers 1, 2) Poincaré has, among other things, discussed the topology of curves defined by ordinary differential equations of a simple character. In a recent course of lectures Hilbert laid considerable stress on the importance of these results and exhibited an elegant method for obtaining them in the case of a differential equation of the form $dy/dx = (cx + dy)/ax + by)$. In the following paper I have shown how the same method can be used for an ordinary differential equation of the second order. My results tally with those of Poincaré insofar as the latter are enumerated; but they are more detailed than his and are, I think, more simply obtained.

Given (1) $d^2y/dx^2 = (dx + ey + f \cdot dy/dx)/(ax + by + c \cdot dy/dx)$, a, b, c, d, e, f real constants. Put (2) $dy/dx = z$ and (1) becomes

$$(3) \qquad dz/dx = (dx + ey + fz)/(ax + by + cz).$$

Now write

$$(4) \quad dx/dt = ax + by + cz, \quad dy/dt = z, \quad dz/dt = dx + ey + fz.$$

Multiply the second and third of these equations by m and n respectively and add,

$$(5) \quad d(x + my + nz)/dt = (a + nd)x + (b + ne)y + (c + m + nf)z.$$

Equate the second member of (5) to $\lambda(x + my + nz)$ and determine m, n and λ accordingly. We find

$$(6, 7) \qquad n = (\lambda - a)/\lambda, \quad m = (bd - ac + e\lambda)/d\lambda,$$

$$(8) \quad \lambda^3 - (a + f)\lambda^2 + (af - cd - e)\lambda - (bd - ae) = 0.$$

Indicate the three values of λ arising from (8) and the corresponding values of m, n by subscripts. Put

$$9) \quad x + m_1 y + n_1 z = \xi, \quad x + m_2 y + n_2 z = \eta, \quad x + m_3 y + n_3 z = \zeta$$

and we obtain as the equivalent of (4)

$$(10) \qquad d\xi/dt = \lambda_1 \xi, \quad d\eta/dt = \lambda_2 \eta, \quad d\zeta/dt = \lambda_3 \zeta$$

or, eliminating t,

$$(11) \; d\eta/d\xi = \kappa \cdot \eta/\xi, \; d\zeta/d\xi = \mu \cdot \zeta/\xi, \text{ where } \kappa = \lambda_2/\lambda_1, \; \mu = \lambda_3/\lambda_1.$$

The problem in hand is to investigate the nature of the critical point (0, 0, 0) in x, y, z space for the equations (4). By virtue of the substitutions (9) this point goes over into (0, 0, 0) in ξ, η, ζ space. There are three principal cases to consider, depending upon the roots of (8). They are: I. The λ_i all real and different; II. Two λ_i conjugate complex; III. Two or more λ_i identical.

I. λ_1, λ_2, λ_3 real and different. The integrals of (11) are

$$(12, 13) \qquad \eta = C\xi^{\kappa}, \quad \zeta = D\xi^{\mu} \quad (C, D \text{ integration constants}).$$

The intersections of these two surfaces constitute ∞^2 integral curves. How are these curves distributed in the immediate vicinity of (0, 0, 0)? How many pass through (0, 0, 0)? An answer to these questions amounts to a characterisation of the critical point. There are three possibilities. They arise according as κ, μ are (i) both > 0, (ii) both < 0, or (iii) unlike in sign.

(i) $\kappa > 0$, $\mu > 0$. By appropriately naming the roots of (8) we can make $\kappa > 1$, $\mu > 1$. Every cylinder (12) passes through the ζ-axis and is tangent to the plane $\eta = 0$. Every cylinder (13) passes through the η-axis and is tangent to the plane $\zeta = 0$. There are therefore ∞^2 integral curves passing through (0, 0, 0), all tangent to one of their number, viz., to the ξ-axis. Hilbert would probably call this singular point a Scheitel-scheitelpunkt.

(ii) $\kappa < 0$, $\mu < 0$. We may write (12), (13) in the form

$$(12, 13) \qquad \eta\xi^{-\kappa} = C, \quad \zeta\xi^{-\mu} = D.$$

The only integral curves passing through (0, 0, 0) now are $\xi = 0 = \eta$, $\eta = 0 = \zeta$, and $\zeta = 0 = \xi$. (12) represents a family of hyperbolic cylinders with generators parallel to the

ζ-axis; (13) a family of hyperbolic cylinders with generators parallel to the η-axis. All the ∞^2 integral curves, barring the three above mentioned, tend to hug the ξ-axis and the $\eta\zeta$-plane. Hilbert would call this singular point a Sattel-sattelpunkt.

(iii) $\kappa > 0$, $\mu < 0$ (or $\kappa < 0$, $\mu > 0$). Write (12), (13) in the form

$$(12,\ 13) \qquad \eta = C\xi^{\kappa}, \quad \zeta\xi^{-\mu} = D.$$

Every cylinder (12) yields two integral curves through (0, 0, 0), viz., the curves of intersection with $\xi = 0$ and with $\zeta = 0$ (these two planes are members of (13)). There are thus ∞^1 integral curves through the origin, all of them lying in one plane, except one, this one being the ζ-axis. Hilbert would probably call this singular point a Scheitel-sattelpunkt mit isoliertem Strahl.

II. λ_1 real; λ_2, λ_3 conjugate complex. The integrals of (10) may now be written

$$(12,\ 14) \qquad \eta = C\xi^{\kappa}, \quad \zeta = D\eta^{\nu}$$

$$(\kappa = \lambda_2/\lambda_1,\ \nu = \lambda_3/\lambda_2,\ C,\ D \text{ arbitrary constants}).$$

If we seek an interpretation of (12), (14) in real space by means of the substitutions

$$\begin{aligned} &\xi = x + m_1 y + n_1 z = X, \\ (15)\quad &\eta = x + m_2 y + n_2 z = x + m_2' y + n_2' z + i(m_2'' y + n_2'' z) = Y + iZ, \\ &\zeta = x + m_3 y + n_3 z = x + m_2' y + n_2' z - i(m_2'' y + n_2'' z) = Y - iZ, \end{aligned}$$

whereby

$$m_2 = m_2' + im_2'', \quad m_3 = m_2' - im_2'', \quad n_2 = n_2' + in_2'', \quad n_3 = n_2' - in_2'',$$

we get from (12)

$$(16) \quad \begin{aligned} Y + iZ &= CX^{\lambda_2/\lambda_1} = CX^{\lambda_2'/\lambda_1} \cdot e^{i(\lambda_2''/\lambda_1)\log X} \\ &= CX^{\lambda_2'/\lambda_1}\{\cos[(\lambda_2''/\lambda_1)\log X] + i\sin[(\lambda_2''/\lambda_1)\log X]\} \\ &\qquad (\lambda_2 = \lambda_2' + i\lambda_2''). \end{aligned}$$

Separating real and imaginary parts in (16) and equating,

$$(17)\qquad Y = CX^{\lambda_2'/\lambda_1} \cos\left[(\lambda_2''/\lambda_1) \log X\right],$$

$$(18)\qquad Z = CX^{\lambda_2'/\lambda_1} \sin\left[(\lambda_2''/\lambda_1) \log X\right].$$

Squaring (17), (18) and adding we obtain

$$(19)\qquad Y^2 + Z^2 = C^2 X^{2\lambda_2'/\lambda_1},$$

a surface, or rather ∞^1 surfaces, of revolution upon which all the ∞^2 integral curves lie. Again, we have from (14) by means of (15),

$$(20)\qquad Y - iZ = D(Y + iZ)^{\nu}.$$

Let us introduce polar coördinates into the Y, Z plane by putting $Y + iZ = \rho e^{i\phi}$. We obtain from (20)

$$(21)\qquad \begin{aligned} \rho &= \text{constant} \cdot e^{i\phi(1+\nu)/(1-\nu)} \\ &= \text{constant} \cdot e^{\phi\lambda_2'/\lambda_2''} = e^{(\lambda_2'/\lambda_2'')(\phi + \text{const.})}, \end{aligned}$$

a family of ∞^1 spiral cylinders winding about the X-axis. The ∞^2 integral curves are thus the intersections of the two sets of surfaces in X, Y, Z space

$$(18)\qquad Z = CX^{\lambda_2'/\lambda_1} \cdot \sin\left[(\lambda_2''/\lambda_1) \log X\right],$$

$$(21)\qquad \rho = e^{(\lambda_2'/\lambda_2'')(\phi + C')} \quad (C' \text{ an arbitrary constant}).$$

The critical point $(0, 0, 0)$ in X, Y, Z space, which is likewise the origin in x, y, z space, might be called a Strudelpunkt, by analogy with the corresponding situation which arises for a differential equation of the first order. The surfaces (18) lie entirely in space $X > 0$, and they all contain the Y-axis as a "stop line" when $\lambda_2'/\lambda_1 > 0$. If $\lambda_2'/\lambda_1 \ngtr 0$, none of the surfaces (18) (except the one for $C = 0$) contains the Y-axis — they cease being defined when X approaches zero. In fact, these cylinders (18) have, along the Y-axis, a definite tangent plane only when $\lambda_2'/\lambda_1 > 1$. In the present case, therefore, there is no integral curve containing $(0, 0, 0)$. If, however, $\lambda_2'/\lambda_1 > 1$, there will be ∞^2 curves which approach $(0, 0, 0)$ asymptotically in space $X > 0$.

For the topological discussion of the equation of the first order there is but one integral equation, viz., the analogon of

(21), and the special assumption $\lambda_2'/\lambda_2'' = 0$ then yields the special class of singular point called Wirbelpunkt, which is completely enclosed by each one of the ∞^1 integral curves $\rho = \text{constant}$. Here, however, in space, no such situation can arise, because (18) loses geometric meaning for $\lambda_2'/\lambda_2'' = 0$. There appears to be nothing in the present problem analogous to the Wirbelpunkt in the plane.

III$_1$. $\lambda_3 = \lambda_2 \neq \lambda_1$. The three equations (10) now shrink to two, and another means of integrating (4) must be devised. Take the three equations

$$d\xi/dt = \lambda_1 \xi, \quad d\eta/dt = \lambda_2 \eta, \quad dz/dt = dx + ey + fz.$$

Assume the second member of the third equation identically equal to $\alpha\xi + \beta\eta + \gamma z$. This necessitates

$$\alpha = \lambda_1 d/(\lambda_1 - \lambda_2), \quad \beta = -\lambda_2 d/(\lambda_1 - \lambda_2), \quad \gamma = \lambda_2,$$

so that, instead of (10), we have as our three equations

$$(22) \quad \begin{aligned} &d\xi/dt = \lambda_1 \xi, \quad d\eta/dt = \lambda_2 \eta, \\ &dz/dt = (\lambda_1 d/(\lambda_1 - \lambda_2))\xi - (\lambda_2 d/(\lambda_1 - \lambda_2))\eta + \lambda_2 z. \end{aligned}$$

Eliminating t,

$$(23) \quad \begin{aligned} &d\xi/d\eta = (\lambda_1/\lambda_2)(\xi/\eta), \\ &dz/d\eta = (\lambda_1 d/\lambda_2(\lambda_1 - \lambda_2))(\xi/\eta) - d/(\lambda_1 - \lambda_2) + z/\eta. \end{aligned}$$

Solving the first of these and making use of the result in the second we obtain as complete integral

$$(12) \qquad \xi = C\eta^{1/\kappa},$$

$$(24) \quad z = (C\lambda_1 d/(\lambda_1 - \lambda_2)^2)\eta^{1/\kappa} - (d/(\lambda_1 - \lambda_2))\eta \log \eta + D\eta$$

(C, D arbitrary).

As before noted, every surface (12) passes through the origin in ξ, η, z space, which is also the origin in x, y, z space. Different possibilities can arise according as κ is > 0 or < 0, and according as d is zero or not.

$d \neq 0$, $\kappa > 0$. Each of the surfaces (24) contains the ξ-axis as a "stop line," since η must be $\geqq 0$; and each of these surfaces is tangent along the ξ-axis to the ξz-plane, since

$dz/d\eta = \infty$ for $\eta = 0$. Hence all the ∞^2 integral curves enter (0, 0, 0) in the half space $\eta \geqq 0$. This singular point is of the sort which Hilbert calls Endpunkt.

$d \neq 0$, $\kappa < 0$. Write (12), (24) in the following form

$$(12) \qquad \xi\eta^{-1/\kappa} = C,$$

$$(24) \quad z\eta^{-1/\kappa} = C\lambda_1 d/(\lambda_1 - \lambda_2)^2 - (d/(\lambda_1 - \lambda_2))\eta^{1-1/\kappa} \cdot \log \eta + D\eta^{1-1/\kappa}.$$

For $C = 0$ (12) gives the planes $\xi = 0$, $\eta = 0$, while (24) gives ∞^1 cylinders in space $\eta \geqq 0$ and having the ξ-axis as "stop line." The intersections of (12) and (24) therefore furnish, for $C = 0$, ∞^1 curves containing (0, 0, 0), all but one of them lying in the half-plane $\xi = 0$ $\eta \geqq 0$ and having (0, 0, 0) as "stop point," while that exceptional one is the ξ-axis itself.

For $C \neq 0$, (12) gives a set of ∞^1 hyperbolic cylinders with generators parallel to the z-axis, while (24) gives (for η small but $\nless 0$) a set of ∞^2 hyperbolic half-cylinders with generators parallel to the ξ-axis. The intersections of (12) and (24), therefore, furnish, for $C \neq 0$, ∞^2 curves all lying in space $\eta > 0$; and none of these contains (0, 0, 0). The singular point for $d \neq 0$, $\kappa < 0$ is thus of a kind distinct from those previously considered. Only one integral curve passes through (0, 0, 0); ∞^1 integral curves have (0, 0, 0) as "stop point;" and ∞^2 integral curves pass by (0, 0, 0) in arbitrary proximity. This singular point might be call Sattel-Endpunkt mit isoliertem Strahl.

$d = 0$, $\kappa \neq 0$. Our integrals now take the form

$$(12,\ 25) \qquad \eta = C\xi^{\kappa}, \quad z = D\eta.$$

We have here no new kind of singular point. If $\kappa > 0$, the origin is a Scheitelpunkt. If $\kappa < 0$, the origin is a Sattelpunkt.

III_{II}. $\lambda_1 = \lambda_2 = \lambda_3 \neq 0$. The three equations (10) now shrink to one, and we must seek still another means of integrating (4). Assume

$$(26) \quad dx/dt = \alpha x + \beta\eta + \gamma z, \quad d\eta/dt = \lambda\eta, \quad dz/dt = \alpha' x + \beta'\eta + \gamma' z,$$

where α, β, γ, α', β', γ' are to be determined so that the equations (26) shall be identities in x, y, z. The appropriate values of these constants are

$$\alpha = \lambda^2(a-\lambda)/(\lambda^2+e), \quad \beta = bd/(\lambda^2+e), \quad \gamma = (c\lambda^2 - b\lambda + ab + ce)/(\lambda^2+e),$$

$$\alpha' = \lambda^2 d/(\lambda^2+e), \quad \beta' = ed/(\lambda^2+e), \quad \gamma' = \lambda(f\lambda + 2e)/(\lambda^2+e).$$

We may now write (26) in the form

$$(27) \qquad \begin{aligned} dx/d\eta &= \alpha x/\lambda\eta + \gamma z/\lambda\eta + \beta/\lambda, \\ dz/d\eta &= \alpha' x/\lambda\eta + \gamma' z/\lambda\eta + \beta'/\lambda. \end{aligned}$$

From this we can pass, by the substitution $u = x + (\lambda - \alpha)z/\alpha'$, to the equivalent pair

$$(28) \qquad \begin{aligned} du/d\eta &= u/\eta + (\lambda^2+e)/\lambda^2, \\ dz/d\eta &= \lambda du/(\lambda^2+e)\eta + z/\eta + ed/\lambda(\lambda^2+e). \end{aligned}$$

The integrals of (28) are

$$(29) \qquad u = \eta \log (C\eta^{1+e/\lambda^2}),$$

$$(30) \quad z = (d/\lambda)[(e+\lambda^2 \log C)\eta \log \eta/(e+\lambda^2) + \tfrac{1}{2}\eta(\log \eta)^2] + D\eta.$$

By all the substitutions above used, the origin has remained unchanged, so that in u, η, z space the critical point is still (0, 0, 0). In (29) η may take negative as well as positive values, but not for the same value of C; so that the z axis is a "stop line" for every surface represented by (29). In (30), however, η must be $\geqq 0$ as long as $d \neq 0$. We must therefore distinguish two possibilities, according as d is, or is not, zero.

$d \neq 0$. All the surfaces (29) will enter the z-axis tangent to the plane $\eta = 0$ except when $1 + e/\lambda^2 = 0$; but this exception may be excluded, since the above transformations would then be impossible. Every surface (30) contains the u-axis as a "stop line" and enters that line tangent to the plane $\eta = 0$. The peculiarity of the singular point (0, 0, 0) in the present case is that every curve of intersection of (29) and (30) not only enters it as a "stop point," but enters it from the same octant. The point might still be called an Endpunkt.

$d = 0$. The surfaces (30) are now a one-parameter set of planes through the u-axis. All the integral curves have (0, 0, 0) as a "stop point," which is therefore to be classed as Endpunkt.

ZÜRICH, *August* 14, 1907.

THE THEORY OF ELECTRICITY.

Theorie der Elektrizität. Von M. ABRAHAM. Zweiter Band: *Elektromagnetische Theorie der Strahlung.* Leipzig, B. G. Teubner, 1905. x + 404 pp.

THE first volume of Abraham's treatise on electricity, which was written as a revision of Föppl's earlier book on the same subject, has already been reviewed in the BULLETIN.[*] At the close of that notice the statement was made that "The second volume will be awaited with impatience." The volume appeared within a few months, and by this time the author would probably feel justified in concluding that however much the reviewer may have been impatient for the appearance of the volume, he was not particularly impatient to review it. This very tardiness will, however, be of no inconsiderable aid in writing a comment on the book.

Abraham's second volume deals with the theory of electrons. It is divided into two parts of which the first treats the field and motion of individual electrons and the second the electromagnetic phenomena in ponderable bodies. Although physical science has taken long strides in the past forty months along the path of electron theory, so that now the electron and its major properties must be considered by all as firmly intrenched facts of physics instead of grudgingly acknowledged theories, and although J. J. Thomson in a recent number of the *Philosophical Magazine* has contributed much in the way of enlightenment to our hitherto very vague notion of the nature of positive electricity, nevertheless comparatively little of that which has been accomplished in addition to what was known at the time of the publication of this volume can as yet be construed as offering very material aid in constructing or revising the mathematical theory of electricity from the electronic point of view. What the electron does in the large — a large which may be measured in small fractions of a $\mu\mu$ — is tolerably well known; but what its characteristics and behavior may be within a distance of one or two 10^{15}ths of a centimeter from its center is still much of an eleusinian mystery.

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[*] Volume 11, pp. 383–387 (April, 1905). The date at the end of the review should be February, 1905, not February, 1904. —We may note that a new edition of this first volume has just appeared.

The uninitiated may not be aware that there are at least four distinct electrons which have reached a considerable mathematical development. The first of these may be called the Larmor electron, inasmuch as the best exposition of its properties is found in Larmor's Aether and Matter. It is a mathematical point endowed with a finite charge of electricity which creates a certain strain or "beknottedness" in the surrounding ether. Of course nobody, Larmor least of all, would maintain that the real physical electron was thus devoid of extension in space. The conception is merely preliminary and is adopted for the purpose of obtaining mathematical simplicity. The second electron may be called the Abraham electron. It is discussed in detail in the book under review and previously to the publication of this book it had already received great development at the hands of the author. Its dimensions are finite and its configuration is spherical with a radius r which probably lies within the limits $10^{-13} < r < 2 \cdot 10^{-13}$. It is rigid. As Lorentz is so nearly the father of all mathematical electron theory, it is perhaps unfair to designate any particular electron by his name. The particular electron, however, which is called the Lorentz electron is also of finite dimensions. When at rest it is spherical, but when animated with a motion of translation it becomes an ellipsoid of revolution with the axis of revolution shortest and directed along the line of motion. If r be the radius of the sphere at rest and if β be the ratio of the velocity of the electron to the velocity of light, the semiaxes of the ellipsoid perpendicular to the line of motion remain equal to r, whereas the axis along the path becomes $r(1-\beta^2)^{\frac{1}{2}}$. This introduces a shrinking along the line of motion and was adopted for the purpose of explaining the troublesome Michelson-Morley experiment. Finally there is the nearly simultaneous creation of Bucherer and Langevin — an electron which shortens in the direction of motion but expands in the perpendicular direction by an amount sufficient to make the volume of the ellipsoid equal to the volume of the original sphere; the new semiaxes are $r(1-\beta^2)^{-\frac{1}{6}}$, $r(1-\beta^2)^{-\frac{1}{6}}$, and $r(1-\beta^2)^{\frac{1}{3}}$.

Fortunately the greater part of the mathematical theory of each of these four electrons is common to the other three. It is only in effects which may be called of the second order that radical differences occur, and even then the differences are not always great. Two very delicate experiments, one the Michelson-Morley experiment already referred to and the other the

Kaufmann experiment, have perhaps been the chief tests of the applicability of the different theories. On the hypothesis that an electron is a charged particle of finite size, the particle in motion will possess an inertia of electric origin which will increase with the velocity and become infinite when the velocity approaches that of light. The increase is very slow at first and scarcely becomes appreciable until the fraction β has passed the the value 0.5. If it be assumed that the entire inertia of the electron is of electric origin — a fact which seems to be clearly indicated by Kaufmann's earlier experiment — it follows that the mass of the electron, whether the transversal or the longitudinal mass, is a subject for purely electromagnetic experiment and theory.

If m denote the transversal mass and m_0 be the value of this mass when $\beta = 0$, the mass for any velocity may be written as $m_0\Phi(\beta)$, where $\Phi(\beta)$ is a function of β which has different expressions according to the different theories, namely,

$$\Phi(\beta) = \frac{3}{4}\frac{1}{\beta^2}\left(\frac{1+\beta^2}{2\beta}\log\frac{1+\beta}{1-\beta} - 1\right) \text{ for Abraham's electron,}$$

$$\Phi(\beta) = (1-\beta^2)^{-\frac{1}{2}} \text{ for Lorentz's electron,}$$

$$\Phi(\beta) = (1-\beta^2)^{-\frac{1}{3}} \text{ for Bucherer's electron.}$$

Since the appearance of Abraham's book, Kaufmann has carried out an extended series of experiments * from which he infers that either the Abraham or the Bucherer electron represents the experimental facts within an error which is less than the experimental error, whereas Lorentz's electron gives calculated values which depart from the observed values by an amount considerably in excess of the experimental errors. Hence, so far as these particular experiments are concerned, it appears that there are still two types of electron to dispute the field.

Another lengthy investigation † which has appeared since Abraham's book is due to Poincaré. Instead of being experimental it is purely mathematical and based on the Michelson-Morley experiment and the principle of relativity. Here the

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* Printed in the *Annalen der Physik*, volume 19 (1906). The range of values of β is about $0.55 < \beta < 0.97$. It is really remarkable that two theories should agree within the small error of two or three per cent. over such a wide range.

† Printed in the *Rendiconti di Palermo*, volume 21 (1906).

theory of groups of transformations does valiant service. The principle of relativity is the postulate that there never can be obtained a method for distinguishing absolute motion; and this certainly accords with past experiments on the subject, which have constantly afforded merely negative results. Poincaré examines the consequences of the assumptions of Abraham, Lorentz, and Bucherer. He finds, among other things, that if an electron is constrained by a relation between its three axes and if no other forces not of electromagnetic origin act upon the electron, then the Bucherer-Langevin hypothesis is the only one admissible. On the other hand this hypothesis does not admit the principle of relativity. Lorentz had previously established this result in a different manner. Poincaré finds also that the only hypothesis which is in accord with the principle of relativity is that of Lorentz; but here it is necessary to introduce forces of order other than electromagnetic, and these forces are derivable from a potential which is proportional to the volume of the electron.

It is needless to remark that in a subject so replete with hypotheses and difficulties as the electron theory there is always a possibility that something has been overlooked and that some day another interpretation may be available which will vitiate present conclusions: but Kaufmann is a particularly careful and habile experimenter and Poincaré is no less an astute mathematician, and it is interesting to combine their conclusions even if to-morrow may call for a revision of the inference. From Kaufmann's experiment let us conclude to throw out the Lorentz electron. Poincaré's work then shows that the principle of relativity cannot subsist. It is certainly more satisfactory philosophically and scientifically to be left with the hope that some time we may be able to distinguish absolute motion than to feel that we shall in nowise be able to do so. Again making an appeal to Poincaré's investigation, we may infer that the Bucherer-Langevin electron is the best to adopt inasmuch as it does not necessitate the introduction of other forces than those of electromagnetic origin and those involved in the relation between the axes of the ellipsoid of revolution. Surely students of electricity will prefer the simple expression of $\Phi(\beta)$ in this system to that given by Abraham, and to any follower of Maxwell the supposition that the volume of the electron is constant must be a source of considerable consolation.

The first chapter of Abraham's book is on the physical and mathematical foundation of the electron theory. The treatment is excellent. The historical notes, the collection of evidence culminating in the irresistible inference of the existence of electrons and their simple relations to problems of electromagnetism, the insistence on the numerical values of the fundamental magnitudes, and the systematic classification of radiation make greatly for the ease and contentment of the reader. No sooner are the fundamental equations set up than the author proceeds to develop the ideas of electromagnetic momentum and moment of momentum. This parallel to ordinary mechanics aids in the comprehension of the text and allows the author to emphasize the differences between the laws of electricity and those of matter. The question of the inequality of action and reaction, which at first caused considerable difficulty to some investigators, is treated with a detail and clearness which leave nothing to be desired. The important results are stated here and throughout the book in spaced type so as to catch the eye. The integration of the equations to obtain the formulas for the potentials is accomplished by a method due to Abraham and based on the familiar method given for a special case in Weber's Particlle Differentialgleichungen. This is the only complicated piece of analysis in the chapter.

The second chapter treats the radiation of waves from a point charge in motion and is consequently of importance in the theory of optics. It will not be amiss to call especial attention to the words point charge. The author takes pains to indicate what portions of his theory are independent of his particular assumption of rigid spherical electrons. This is of high value to the reader, for whom it is a matter of importance to have those portions of the work which would be true on any of the current hypotheses concerning electrons separated from the consequences due to Abraham's special electron. This chapter, then, is concerned with what happens at a considerable distance from the electron. The model for a radiating source of light is the electric dipole, consisting of a fixed positive charge and a moving negative charge. The loss of energy through radiation is computed, and not merely computed as a formula but actually turned into numbers from available experimental data. The emphasis laid upon the order of numerical magnitude of the quantities involved in a calculation will almost inevitably serve to differentiate a true mathematical physicist from a mathematician who merely looks to physics for problems to which to apply his analysis. In this chapter the Zeeman effect naturally comes in for a detailed discussion. The field due to an electron moving uniformly or with an acceleration is taken up and applied to the theory of a moving source of light. The questions of longitudinal and transverse vibrations are handled separately, and the matter of the reaction of the radiation on the source is not overlooked. It is a happy idea of the author's not to be in any haste in passing over these fundamental points.

The mechanics of electrons is the subject of the third chapter. There is an introduction in which the reasons for assuming that the mass of the electron is wholly of electromagnetic origin are outlined and in which the relation of the electromagnetic cosmos to the mechanical cosmos discussed by Hertz in his Mechanics is set forth. The author then gives a detailed argument in favor of his rigid spherical electron. His aim is to construct the world on a purely electromagnetic basis, and the assumption of any other hypothesis concerning the electron appears to him to necessitate the introduction of enormous elastic forces of non-electromagnetic origin. This seems scarcely conclusive, especially in view of Poincaré's recent memoir. It is well known that when a charged sphere is in motion the lines of force, alias the Faraday tubes, are drawn toward a plane through the center of the sphere and perpendicular to the path. Why this might not quite naturally result in the Bucherer-Langevin deformation is hard to see. In fact it would seem that when an electron is in motion it would require enormous forces of rigidity which were not of electromagnetic origin to preserve the spherical shape of the electron. If the author were to reprint his book now, he would doubtless give this question a more thorough treatment.

In this chapter on the mechanics of electrons Abraham again keeps close enough to ordinary mechanics to speak of momentum, moment of momentum, moment of inertia, and the lagrangian function. This part of the work is as interesting as any and is highly to be recommended for its pedagogic excellence, especially in view of the fact that when the book was written it had many persons to convince as well as to instruct. The matter of electromagnetic mass, whether transversal or longitudinal, is expounded in all detail and is set into relation with the lagrangian function. The Lorentz electron is treated and its relation to the author's is indicated. The discussion ends with the statement that Lorentz had shown that his electron satisfied the (earlier) Kaufmann experiments about as well as Abraham's and with the expression of the hope that further refinements of the experiment would serve to differentiate between the two. That hope was soon to be realized in favor of the author. Unfortunately no word is mentioned concerning Bucherer's electron. This is a serious omission, although perhaps unavoidable, inasmuch as Bucherer's book * had appeared only some six or eight months before Abraham's. The chapter concludes with some interesting though partly speculative investigations such as the discontinous motion of electrons (useful in the explanation of Röntgen rays), the force in the interior of an electron, and the uniform motion of an electron with a velocity greater than that of light.

The author then comes to the second part of his work, that on electromagnetic phenomena in ponderable bodies. This he divides into two chapters, the first on stationary bodies, the second on moving bodies. After a preliminary discussion of what is to be understood by the term physically small, he introduces the method of averages to derive the ordinary equations of the electromagnetic field in bodies at rest. The dispersion of electromagnetic waves and the connection with the index of refraction, the magnetic rotation of the plane of polarization, the question of magnetization, and electric conduction in metals are treated from the point of view of electron theory. The name of Drude occurs frequently in these pages, but we do not find that of J. J. Thomson. To be sure, a great deal of Thomson's work † in this particular direction had not appeared at the time of writing and considerable more of it might have been thought to be still in too speculative a state and insufficiently capable of presentation in a form assimilable with the rest of the book; this, however, was hardly true of all of it. Other omissions, such as the theory of optical rotation in crystals and solutions and the explanation of the Peltier effect, may be explained by the simple statement that even now electrons have not enabled us to account for all electrical phenomena. As a general criticism, though not a severe one, we may say

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* Mathematische Einführung in die Elektronentheorie. B. G. Teubner, 1904, 148 pp. This is perhaps the best short account of an electron theory. On account of its brevity it is more exclusively mathematical and consequently of greater average difficulty than the work under review.

† We may mention his book Electricity and Matter, Charles Scribner's Sons, 1904; 162 pp. Also the extremely recent Corpuscular Theory of Matter, Constable, 1907; 180 pp. The latter we have not yet seen.

that it would have added to the interest of the book if the author had gone more into detail from time to time as regards the things which were yet to be accounted for by the theory instead of confining himself so exclusively to matters which he actually was in a position to analyze. In conclusion to the chapter there is a lengthy discussion of the mathematics of wireless telegraphy.

The concluding chapter on moving bodies commences with the careful and critical derivation of the equations of the field followed by a discussion of Fizeau's experiment. The crucial nature of this experiment for deciding between Lorentz's optics and Hertz's is brought out. There follow sections on the pressure of radiation on surfaces in motion, whether they be black or reflecting. It would have been possible to add somewhat to the interest of this question by entering upon Poynting's applications of the results to cosmical speculations. In the section on the temperature of radiation we find the laws of Kirchhoff and Wien. Here the author, as in so many other places, goes into the matter numerically. Next follows the treatment of the Michelson-Morley experiment and its crucial evidence against a stagnant ether or in favor of a contraction along the path. This leads to a presentation of the optics of Lorentz and Cohn; and with that the book comes to a close except for the extensive index.

From what has been said it cannot fail to appear that we have in this treatise a work which deals with the most fundamental questions of physics and sets them into relation with the latest developments of theory and experiment. In only a few places is the analysis complicated, and everywhere there is an abundance of physical data which are frequently worked out to their numerical consequences. To an unusually large extent the book represents the work of individual investigation on the part of the author. It could not have been written as a compilation from the accomplishments of others. Whether the Abraham electron shall persist or be cast aside, the greater portion of the present volume will remain, and most of the rest will have to be changed but little. Clearly those who impatiently awaited the appearance of this second volume have not been disappointed in their expectations of it.

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, MASS., *December*, 1907.

NOTES.

THE opening (January) number of volume 9 of the *Transactions of the* AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Certain periodic orbits of k finite bodies revolving about a relatively large central mass," by F. L. GRIFFIN; "Further note on Maclaurin's spheroid," by G. H. DARWIN; "Potential functions on the boundary of their regions of definition" and "Double distributions and the Dirichlet problem," by O. D. KELLOGG; "Groups defined by the orders of two generators and the order of their commutator," by G. A. MILLER; "Projective differential geometry of curved surfaces" (second memoir), by E. J. WILCZYNSKI.

THE opening (January) number of volume 30 of the *American Journal of Mathematics* contains the following papers: "Plane quintic curves which possess a group of linear transformations" and "On birational transformations of curves of high genus," by V. SNYDER; "Surfaces with the same spherical representation of their lines of curvature as spherical surfaces," by L. P. EISENHART; "On the factoring of composite hypercomplex number systems," by H. B. LEONARD; "A new method in geometry," by E. LASKER; "Groups generated by n operators each of which is the product of $n-1$ remaining ones," by G. A. MILLER.

AT the meeting of the London mathematical society held on December 12 the following papers were read: By S. T. SHOVELTON, "A formula in finite differences and its application to mechanical quadrature"; by A. E. H. LOVE, "Weierstrass's excess-function in the calculus of variations."

AT the Dresden meeting of the Deutsche Mathematiker Vereinigung Professor FELIX KLEIN was elected president for the current year, and also named official representative of the society at the fourth international congress of mathematicians, to be held in Rome in April. The present volume (17) of the *Jahresbericht* will have different pagination for the memoirs, from the part devoted to communications, notes, etc. A committee consisting of Mr. ACKERMANN-TEUBNER and Professor GUTZMER was appointed to consider ways and means of making new mathematical books published outside of Germany more accessible. A committee was appointed to cooperate with the Swiss committee to consider the publication of the works of Euler. The committee on methods of instruction will be augmented by inviting two representatives from the various academies and societies that wish to cooperate in the movement of reorganizing mathematical instruction in the secondary schools. The next meeting of the Vereinigung will be held in Cologne in September 1908. It now numbers 703 members.

AT the annual public meeting of the Paris academy of sciences, held on December 16, 1907, the following prizes were awarded for memoirs in pure and applied mathematics: Francoeur prize (fr. 1000) to E. LEMOINE, for his contributions to geometry; Bordin prize (fr. 3000) divided between F. ENRIQUES and F. SEVERI, for the solution of the problem announced in the BULLETIN, volume 11, page 338; Vaillant prize (fr. 4000) divided among J. HADAMARD, A. KORN, G. LAURACELLA and T. BOGGIO, for contributions to the theory of the equilibrium of supported elastic plates; Montyon prize (fr. 700) to M. CUENOT, for his analysis of the flexibility of railroad rails; Poncelet prize (fr. 2000) to Col. RENARD for his researches in mechanics; Lalande prize (fr. 540) to T. LEWIS, for his contributions to astronomy; Valz prize (fr. 460) to M. GIACOBINI for his work in astronomy; Pontécoulant prize (fr. 700) to M. GAILLOT for celestial mechanics; Petit d'Ormay prize (fr. 1000) to P. DUHEM for his work in physical chemistry; Laplace prize (complete works of Laplace) to L. DAUM for his work in mechanics.

The following university courses in mathematics are announced:

OXFORD UNIVERSITY (*Hilary Term*). — By Professor W. ESSON: Comparison of analytic and synthetic methods in the theory of conics, two hours; Synthetic geometry of cubics, one hour. — By Professor E. B. ELLIOT: Elements of elliptic functions, two hours; Theory of numbers, II, one hour. — By Professor H. H. TURNER: Elementary mathematical astronomy, two hours. — By Professor A. E. H. LOVE: Theory of the potential, two hours; Elements of the calculus, two hours. — By Mr. P. J. KIRKBY; Higher plane curves, two hours. — By Mr. A. L. DIXON: Calculus of finite differences, one hour.

— By Mr. J. E. CAMPBELL: Differential equations, II., two hours. — By Mr. C. H. THOMPSON: Dynamics of a particle, two hours. — By Mr. H. T. GERRANS: Hydrodynamics, two hours — By Mr. C. E. HASELFOOT: Geometric optics, two hours. — By Mr. J. W. RUSSELL: Determinants, two hours. — By Mr. A. L. PEDDER: Trigonometry, one hour. — By Mr. A. E. JOLLIFFE: Analytic geometry, II, two hours. — By Mr. R. F. McNEILE: Integral calculus, two hours. — By Mr. E. H. HAYES: Mechanics, three hours. — By Mr. C. H. SAMPSON: Solid geometry, two hours.

THE Royal Belgian academy of sciences has awarded a prize of 800 francs (prize in geometry for 1907) to Professor G. BORDIGA, of the University of Padua; it also awarded a prize of 600 francs to Professor U. PERAZZO, of the military academy of Turin.

PROFESSOR FELIX KLEIN, of the University of Göttingen, has been appointed representative of the Universities in the upper house of the German parliament.

PROFESSOR D. HILBERT, of the University of Göttingen, has been decorated with the Maximilian order of Bavaria.

DR. A. H. BUCHERER, of the University of Bonn, has been promoted to an associate professorship of mathematics.

DR. P. MUTH, although not connected with any institution of learning, has received the title of professor, in recognition of his mathematical writings.

DR. P. BURGATTI, of the University of Rome, has been appointed associate professor of theoretical mechanics at the University of Messina.

PROFESSOR E. ALMANSI, of the University of Pavia, has been promoted to a full professorship.

PROFESSOR R. MARCOLONGO, of the University of Messina, has been appointed professor of theoretic mechanics at the University of Naples.

DR. G. PICCIATI, of the University of Padua, has been appointed associate professor of mathematical physics at the University of Bologna.

PROFESSOR O. TEDONE, of the University of Genoa, has been promoted to a full professorship of mechanics.

Professor G. Vivanti, of the University of Messina, has been appointed professor of the calculus at the University of Pavia.

Dr. E. Bianchi has been appointed docent in mathematical astronomy at the University of Rome.

Dr. L. Sinigallia has been appointed docent in the calculus at the University of Pavia.

Dr. A. Denizot has been appointed docent in mathematics at the technical school of Lemberg.

Dr. F. Bernstein, of the University of Halle, has been appointed associate professor of mathematics, with particular reference to insurance, at the University of Göttingen.

Dr. F. Graefe, associate professor of mathematics at the technical school of Darmstadt, has been given the title of honorary professor of mathematics.

Professor G. Kowalewski, of the University of Bonn, has declined the call to the mining academy at Clausthal.

Dr. Pflüger, of the University of Bonn, has been appointed associate professor of mathematical physics.

Dr. M. Winkelmann has been appointed docent in mathematics at the technical school of Karlsruhe.

Professor G. Blazek, of the technical school at Prague, has retired.

Dr. H. F. Baker has been appointed chairman of the examiners of the mathematical tripos, part II, for 1908.

Professor Simon Newcomb has been elected foreign member of the academy of sciences at Göttingen.

Dr. G. W. Hill has been elected honorary member of the Royal society of Edinburgh.

Professor E. O. Lovett, of Princeton University, has resigned his professorship to accept the presidency of the Rice Institute, Houston, Texas.

Mr. G. E. Ramsdell has been appointed professor of mathematics at Bates College, Lewiston, Maine, successor to the late Professor J. H. Rand, who died November 7, 1907.

Dr. L. Cohen has been appointed assistant professor of mathematics at the George Washington University.

Sir William Thomson (Lord Kelvin) died at Glasgow, December 17, 1907. He was born at Belfast, June 25, 1824, and graduated from St. Peter's, Cambridge, in 1845, being first Smith's prizeman and second wrangler as well as oarsman in the victorious Cambridge crew. The next year he was appointed professor of natural philosophy at the University of Glasgow, which post he held for over 50 years. His scientific assistance made the Atlantic cable a success; for this achievement he was knighted in 1866. His improvement of the compass, of the method of measuring electricity and of predicting the level of the tides, were all epoch-making advances. He was elected to membership in nearly all the learned societies of Europe, and elevated to the peerage in 1892. His principal writings are "A treatise on natural philosophy" (with Professor Tait), "On heat and electricity," "Paper on electrostatics and magnetism," "Mathematical and physical papers" (three volumes), "Popular lectures and addresses" (three volumes), and the "Baltimore lectures." He was buried in Westminster Abbey.

Dr. Asaph Hall died November 22, 1907. Born at Goshen, Connecticut, October 15, 1829, he was appointed assistant at the Harvard observatory in 1857, Naval observer in 1862 and professor of mathematics in the Navy from 1863 to 1891, when he retired under the age limit. He was then appointed professor of mathematical astronomy at Harvard University, which position he held until 1901. Dr. Hall was a prolific writer, and his investigations won for him numerous medals and membership in most of the learned societies of America and of Europe. Among his most important discoveries may be mentioned the period of Saturn, two new satellites of Mars, the motion of the line of apsides of Hyperion, and important contributions to the theory and data of stellar parallax.

Professor C. A. Young, emeritus professor of astronomy at Princeton University, died January 4, at the age of 73 years.

Dr. A. Lévy, professor of mathematics at the municipal industral school of Paris, died December 28, at the age of 64 years.

Professor T. Barker, formerly professor of mathematics at Owens College, Manchester, England, died November 20. He bequeathed about £40,000 to Victoria University, to establish professorships and scholarships in mathematics and in botany.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BACHMANN (P.). Grundlehren der neueren Zahlentheorie. (Sammlung Schubert, LIII). Leipzig, Göschen, 1907. 8vo. 11 + 271 pp. Cloth. M. 6.50

BAILEY (F. H.). See WOODS (F. S.).

BENNECKE (F.). Eine konforme Abbildung als zweidimensionale Logarithmentafel zur Rechnung mit komplexen Zahlen. Festschrift des königlichen Victoria-Gymnasiums zur 300-jährigen Jubelfeier des königlichen Joachimthalschen Gymnasiums zu Berlin. Berlin, Salle, 1907. 7 pp. M. 2.00

BRYAN (G. H.) and PINKERTON (R. H.). The elements of the geometry of the conic. London, Dent, 1907. 8vo. 282 pp. 3s. 6d.

CAMPBELL (D. F.). A short course in differential equations. Enlarged edition. New York, Macmillan, 1907. 12mo. 7 + 123 pp. Cloth. $0.90

CRAMER (F. H.). Ueber die Erniedrigung des Geschlechtes Abelscher Integrale, insbesondere elliptischer und hyperelliptischer, durch Transformation. Nürnberg, 1907. 8vo. 36 pp. M. 0.60

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Vol: III: Geometrie, redigiert von W. F. Meyer. Teil 1, Heft 2: G. Fano, Gegensatz von synthetischer und analytischer Geometrie in seiner historischen Entwicklung im XIX. Jahrhundert; G. Fano, Kontinuierliche geometrische Gruppen; die Gruppentheorie als geometrisches Einteilungsprinzip. Leipzig, Teubner, 1907. 8vo. Pp. 221–388.

——. Vol. IV: Mechanik, redigiert von F. Klein und C. H. Müller. Teil 2 II, Heft 2: O. Tedone und A. Timpe, Spezielle Ausführungen zur Statik elastischer Körper; H. Lamb, Schwingungen elastischer Körper, insbesondere Akustik. Leipzig, Teubner, 1907. 8vo. Pp. 125–310.

FANO (G.). See ENCYKLOPÄDIE.

GUNTHER (C. O.). Integration by trigonometric and imaginary substitution. With an introduction by J. B. Webb. New York, Van Nostrand, 1907. 8vo. 85 pp. Cloth. $1.25

KLEIN (F.), LAMB (H.). See ENCYKLOPÄDIE.

LERCH (L.). Ueber Zentrum und Achse polarreziproker Dreiecke. Tarnowitz, 1907. 4to. 16 pp. M. 1.50

MARKOFF (A.). See TCHEBYCHEFF (P. L.).

MEYER (W. F.), MÜLLER (C. H.). See ENCYKLOPÄDIE.

PINKERTON (R. H.). See BRYAN (G. H.).

POUHKA (K. A.). Ueber die Cauchy'schen reziproken Funktionen und deren Anwendung auf die analytische Fortsetzung der Taylor'schen Reihe. Helsingfors, 1907. 4to. 4 + 49 pp. M. 2.50

SONIN (N.). See TCHEBYCHEFF (P. L.).

TCHEBYCHEFF (P. L.). Oeuvres. Publiées par les soins de A. Markoff et N. Sonin. Vol. II. St. Petersburg, 1907. 4to. 20 + 736 pp. M. 17.50

TEDONE (O.), TIMPE (A.). See ENCYKLOPÄDIE.

WEBB (J. B.). See GUNTHER (C. O.).

WOODS (F. S.) and BAILEY (F. H.). A course in mathematics, for students of engineering and applied science. Vol. I: Algebraic equations, functions of one variable, analytic geometry, differential calculus. Boston, Ginn, 1907. 8vo. 12 + 385 pp. Cloth. $2.25

II. ELEMENTARY MATHEMATICS.

BALL (W. W. R.). Récréations mathématiques et problèmes des temps anciens et modernes. 2e édition française, traduite d'après la 4e édition anglaise et enrichie de nombreuses additions par J. Fitz-Patrick. 1re partie : Arithmétique, algèbre et théorie des nombres. Paris, Hermann, 1907. 8vo. 362 pp. Fr. 5.00

BAUDOIN (P.). See BOURLET (C.).

BOURLET (C.). Précis d'algèbre, rédigé conformément aux programmes du 27 juillet 1905. Corrigés des 573 exercices et problèmes, avec la collaboration de J. Hulot. 2e édition, revue. Paris, Hachette, 1907. 16mo. 507 pp. Fr. 3.50

BOURLET (C.) et BAUDOIN (P.). Cours abrégé de géométrie, publié avec de nombreux exercices théoriques et pratiques et des applications au dessin géométrique. I: Géométrie plane. Paris, Hachette, 1907. 16mo. 8 + 408 pp. Fr. 2.50

BÜRKLEN (O.). Lehrbuch der ebenen Trigonometrie mit Beispielen und 280 Uebungsaufgaben für höhere Lehranstalten und zum Selbstunterricht. Neue unveränderte Ausgabe. Stuttgart, Kohlhammer, 1907. 8vo. 10 + 122 pp. Cloth. M. 1.50

CRELLE (A. L.). Rechentafeln, welche alles Multiplizieren und Dividieren mit Zahlen unter 1000 ganz ersparen, bei grösseren Zahlen aber die Rechnung erleichtern und sicherer machen. Neue Ausgabe, besorgt von O. Seeliger. Mit Tafeln der Quadrat- und Kubikzahlen von 1–1000. Berlin, Reimer, 1907. 7 + 501 pp. Cloth. M. 15.00

DIEKMANN. See HEILERMANN.

FITZ-PATRICK (J.). See BALL (W. W. R.).

FORNIER Y PADILLA (F.). Nociones de geometría elemental. 2a edición. Barcelona, Carbonell, 1907. 151 pp.

GORE (J. H.). Elements of plane and spherical trigonometry ; also six-place logarithmic tables. New York, Putnam, 1907. 8vo. 6 + 122 + 17 + 62 pp. Cloth. $1.20
——. Without tables. $1.00
——. Tables bound separately. $0.60

HEILERMANN UND DIEKMANN'S Lehr- und Uebungsbuch für den Unterricht in der Algebra an den höheren Schulen. Neu bearbeitet von K. Knops. 2ter Teil. 6te Auflage. Essen, Baedeker, 1907. 8vo. 5 + 214 pp. Cloth. M. 2.80
——. 1ter Teil. 12te Auflage. Auflösungen der Aufgaben. Essen, Baedeker, 1907. 8vo. 58 pp. M. 1.00

HULOT (J.). See BOURLET (C.).

Knops (K.). See Heilermann.

Lange (J.). Synthetische Geometrie der Kegelschnitte nebst Uebungsaufgaben für die Prima höherer Lehranstalten. 3te Auflage, besorgt von P. Zühlke. Berlin, Müller, 1908. 8vo. 68 pp. M. 1.50

Mateo de Iraola (E.). Curso elemental de algebra. 4a edición. Sevilla, Diaz, 1907. 142 pp.

Model answers to second stage mathematics papers. From 1903 to 1907. (For candidates for board of education examination in mathematics.) London, Clive, 1907. 8vo. 80 pp. 6d.

Murray (D. A.). Essentials of trigonometry; including solution of triangles, measurement of areas, heights and distances, the use of logarithms, plotting of graphs, and finding the slopes of curves, with four-place tables. New York, Longmans, 1907. 8vo. 10 + 113 pp. Cloth. $0.80

Peter (F. H.). Anleitung zur Behandlung angewandter Gleichungen. Wien, 1907. 8vo. 36 pp. M. 0.60

Schmehl (C.). Lehrbuch der Arithmetik und Algebra, nebst einer Aufgaben-Sammlung. 1ter Teil. Für die 6-klassigen höheren Lehranstalten und die Klassen Untertertia bis Untersekunda der Vollanstalten. Giessen, Roth, 1908. 8vo. 8 + 391 pp. M. 3.20

Seeliger (O.). See Crelle (A. L.).

Treutlein (P.). Mathematische Aufgaben aus den Reifeprüfungen der badischen Mittelschulen (Gymnasien, Realgymnasien, Oberrealschulen). 2ter Teil. Auflösungen. Leipzig, Teubner, 1907. 8vo. 4 + 75 pp. M. 2.00

Vintéjoux (F.). Eléments d'arithmétique, de géométrie et d'algèbre. 7e édition, revue. Paris, Hachette, 1907. 16mo. 7 + 576 pp. Fr. 2.50

Zühlke (P.). See Lange (J.).

III. APPLIED MATHEMATICS.

Abraham (M.) und Föppl (A.). Theorie der Elektrizität. Vol. I: Einführung in die Maxwellsche Theorie der Elektrizität. Mit einem einleitenden Abschnitte über das Rechnen mit Vektorgrössen in der Physik. 3te, vollständig umgearbeitete Auflage. Leipzig, Teubner, 1907. 8vo. 18 + 460 pp. Cloth. M. 12.00

Blank (C. L.). See Burckhardt (W.).

Borchardt (W. G.). Elementary statics. London, 1907. 8vo. 406 pp. Cloth. 3s. 6d. With answers, 4s. 6d.

Burckhardt (W.) und Blank (C. L.). Mathematische Unterrichtsbriefe für das Selbststudium Erwachsener. Mit besonderer Berücksichtigung der angewandten Mathematik. 1ter Kurs. 4te verbesserte und vermehrte Auflage. Jena, Thüringer Verlagsanstalt, 1907. 8vo. Pp. 1–32. M. 0.60

Clamwell (W.). Elementos de mecanica. Versión española. Edición ilustrada. Barcelona, 1907. 135 pp. P. 1.00

Fenkner (H.). Arithmetische Aufgaben, unter besonderer Berücksichtigung von Anwendungen aus dem Gebiete der Geometrie, Physik und Chemie. Ausgabe A, für Gymnasien, Realgymnasien und Oberrealschulen. Teil IIb: Pensum der Prima. 2te, umgearbeitete Auflage. Berlin, 1907. 8vo. 219 pp. M. 2.60

Föppl (A.). See Abraham (M.).

Laar (J. J. von). Lehrbuch der theoretischen Elektrochemie auf thermodynamischer Grundlage. Leipzig, Engelmann, 1907. 8vo. 12 + 307 pp. Cloth. M. 7.20

Lony (G.). Ueber die beim Nachzeichnen von Streckenteilen auftretenden Grössenfehler. Hamburg, 1907. 4to. 33 pp. M. 1.50

Lorentz (H. A.). Abhandlungen über theoretische Physik. Vol. I. Leipzig, Teubner, 1907. 8vo. 4 + 489 pp. Cloth. M. 17.00

Ortu-Carboni (S.). Matematica finanziaria. Operazioni a breve ed a lunga scadenza (sconti, depositi, rendite, ecc.). Milano, 1907. 8vo. 559 pp. M. 10.00

Powell (R. C.). Tables for engineering calculations. 2d edition, revised and enlarged. Berkeley, Cal., Powell, 1907. 12mo. 8 + 195 pp. Limp leather. $2.00

Schill (R.). Maturitätsaufgaben aus der darstellenden Geometrie nebst vollständigen Lösungen. Für die oberen Klassen der Realschulen und verwandter Anstalten sowie für das Selbststudium zusammengestellt und gelöst. 3ter Teil: Darstellung von Durchdringungen, Rotationskörpern, Schattenkonstruktionen, sowie von dem Wichtigsten aus der Linearperspektive. Wien, Deuticke, 1908. 8vo. 6 + 108 pp. M. 3.00

Schürnbrand (L.). Graphische Tabellen zur Berechnung von Kreisquerschnitten auf Drehung und Biegung sowie von Rechteckquerschnitten auf Biegugg, für alle vorkommenden Momente und zulässigen Spannungen berechnet und entworfen. Wiesbaden, Kreidel, 1908. 11 pp. with 28 tables. Cloth. M. 5.00

Shepard (W. K.). Problems in strength of materials. (Mathematical texts, edited by P. F. Smith.) Boston, Ginn, 1907. 8vo. 7 + 70 pp. Cloth. $1.25

Tétrel (A.). Calculs et problèmes d'électricité théorique et industrielle (Ecole supérieure d'électricité; License; Agrégation; Certificat de physique générale; Ecole normale supérieure; Ecole polytechnique, etc.). Paris, Tétrel, 1907. 8vo. 192 pp.

Book No ......

THE FOURTEENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE fourteenth annual meeting of the Society was held in New York City on Friday and Saturday, December 27–28, 1907. Two sessions were held on Friday and one on Saturday morning. On Friday evening about thirty members gathered at the informal dinner and smoker which has long been a feature of the Society's meetings. The attendance at the three sessions included the following fifty-three members:

Professor Joseph Allen, Professor William Beebe, Mr. William Betz, Professor G. A. Bliss, Professor C. L. Bouton, Professor Joseph Bowden, Professor E. W. Brown, Dr. Emily Coddington, Professor F. N. Cole, Professor T. C. Esty, Professor A. S. Gale, Dr. F. L. Griffin, Professor C. O. Gunther, Professor J. G. Hardy, Professor H. E. Hawkes, Dr. L. I. Hewes, Mr. H. R. Higley, Dr. A. A. Himowich, Dr. J. G. Hun, Professor W. H. Jackson, Mr. S. A. Joffe, Professor C. J. Keyser, Professor P. A. Lambert, Mr. W. D. Lambert, Professor W. W. Landis, Dr. N. J. Lennes, Dr. G. H. Ling, Dr. W. R. Longley, Mr. E. B. Lytle, Professor James Maclay, Professor Max Mason, Mr. C. N. Moore, Dr. R. L. Moore, Professor Frank Morley, Professor Richard Morris, Professor G. D. Olds, Professor Anna H. Palmié, Dr. H. B. Phillips, Professor James Pierpont, Miss Amy Rayson, Mr. H. W. Reddick, Professor R. G. D. Richardson, Professor P. F. Smith, Professor J. H. Tanner, Mr. C. A. Toussaint, Mr. M. O. Tripp, Professor H. W. Tyler, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Professor E. B. Wilson, Professor J. E. Wright, Professor J. W. Young.

President H. S. White presided at the morning sessions. Vice-President P. F. Smith took the chair on Friday afternoon. The Council announced the election of the following persons to membership in the Society: Mr. Charles Ammerman, McKinley High School, St. Louis, Mo.; Dr. C. S. Atchison, Williams College; Mr. B. H. Camp, Wesleyan University; Professor W. M. Carruth, Hamilton College; Mr. G. R. Clements, Williams College; Professor Julia T. Colpitts, Iowa State College; Professor J. N. Ivey, Tulane University; Professor W. H. Jackson, Haverford College; Mr. W. C. Krathwohl, Columbia University; Professor Murray Macneill, Dalhousie University; Mr. C. N. Moore, Harvard University; Professor Maria M. Roberts, Iowa State College; Mr. E. W. Sheldon, Yale University. Seven applications for membership in the Society were received.

In response to the invitation of Professor E. J. Townsend, it was decided to hold the next summer meeting of the Society at the University of Illinois. The date of the next annual meeting was fixed as Tuesday and Wednesday, December 29–30.

The reports of the Treasurer and Auditing Committee will appear in the Annual Register. The membership of the Society has increased during the past year from 554 to 580, including at present 52 life members. The number of papers presented at all meetings during the year was 166. The total attendance of members was 398. The Treasurer's report shows a balance of $6,081.82, of which $2,969.40 is credited to the life membership fund. Sales of the Society's publications during the year amounted to $1,534.

Several important changes have occurred in the past year in the administrative and editorial offices. Dr. Dennett retired in February from the Treasurer's office, and was succeeded by Professor Tanner. Professor Bôcher has succeeded Professor Moore as editor in chief of the *Transactions*. At the close of the year Professor Alexander Ziwet retired from the Committee of Publication, of which he had been a most valued member since 1893. The vacancy in the Committee was filled by the election of Professor Virgil Snyder.

At the annual election, which closed on Saturday morning, the following officers and members of the Council were chosen:

Vice-Presidents,	Professor JAMES HARKNESS, Professor G. A. MILLER.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor D. E. SMITH,
Professor VIRGIL SNYDER.

Members of the Council to serve until December, 1910,

Professor R. E. ALLARDICE,	Professor G. D. OLDS,
Professor MAX MASON,	Professor M. B. PORTER.

The following papers were read at this meeting:

(1) Dr. F. L. GRIFFIN: "Certain families of central orbits with a constant apsidal angle."

(2) Dr. F. L. GRIFFIN: "On the non-existence of certain types of periodic solutions in the problem of three bodies."

(3) Professor E. O. LOVETT: "On a problem in mechanics. Note II."

(4) Professor E. O. LOVETT: "On a class of periodic solutions in the problem of four bodies."

(5) Professors OSWALD VEBLEN and J. W. YOUNG: "A set of assumptions for projective geometry."

(6) Professor FRANK MORLEY: "The transformation of a Clifford configuration into itself."

(7) Professor R. G. D. RICHARDSON: "Lebesgue improper integrals."

(8) Professor E. W. BROWN: "The motion of the moon relative to a moving plane of reference."

(9) Professor E. W. BROWN: "The development of the infinite determinant."

(10) Dr. ARTHUR RANUM: "Groups of singular matrices."

(11) Professor J. G. HARDY: "Curves in a space of n dimensions."

(12) Dr. A. B. COBLE: "A reduction of the problem of solving a quintic to Klein's 'problem of the A's' by means of invariant theory."

(13) Dr. L. I. HEWES: "The necessary and sufficient condition that an ordinary differential equation shall admit a conformal group."

(14) Professor J. I. HUTCHINSON: "Hermitian forms with zero determinant."

(15) Professor E. B. WILSON: "On the uniform rotation of a homogeneous chain about a vertical axis."

(16) Professor E. B. WILSON: "On the theory of double products and strains in hyperspace.

(17) Professor MAX MASON: "Note on implicit functions."

(18) Professor H. E. HAWKES: "On the equivalence of families of bilinear forms."

(19) Professor G. A. BLISS: "An existence theorem for a partial differential equation of the first order which is non-analytic."

In the absence of the authors Professor Hutchinson's paper was presented by Professor J. W. Young, and the papers of

Professor Lovett, Dr. Ranum, and Dr. Coble were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Bertrand has discussed laws of central force for which every orbit is a closed curve.* In reality, however, he proved a far more general theorem than he stated, viz., that there are only two laws for which all the trajectories have a constant angle Θ. In Dr. Griffin's first paper, after noting this fact, consideration is given to those laws for which Θ is constant in certain families of orbits. It is shown that, while (for $n \neq 1, 2$) there is no law for which $\Theta = \pi/n$ in all the orbits, yet (for n arbitrary, rational or irrational) there exist laws of force for which $\Theta = \pi/n$ in a double infinitude of orbits, viz., in those families, the average of the nth powers of the apsidal distances of whose orbits is an arbitrary constant.

2. In a recent paper † Professor Moulton has proved the existence of a certain class of periodic plane orbits of three bodies. In this note Dr. Griffin shows that of two types closely related to these there are no orbits in three dimensions. The physical reasons for this non-existence are pointed out.

3. Professor Lovett considers the most general case of a problem studied in a paper presented to the Society at its last summer meeting (See BULLETIN, volume 14 (1907), page 57). Two differential systems F and G are to be integrated. The system F in its most general form is integrated, and the complete integration of G is achieved for those cases in which the conditions of integrability of F are realized.

As to the parameters, they must satisfy but two of the relations found in the former paper, namely $\alpha : \gamma : \delta = 9 : -45 : 40$. As regards the forces, we have the curious result that there appear no other forms than those originally found by Darboux and Halphen in the case of conic trajectories.

4. In a recent number of the *Annali di Matematica* Mr. G. Pavanini ‡ has constructed a new category of periodic solutions

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* *Comptes rendus*, vol. 77, pp. 849–53.

† *Transactions*, vol. 7, pp. 537–577.

‡ Pavanini, "Sopra una nuova categoria di soluzioni periodiche nel problema dei tre corpi," *Annali di Matematica*, 3d series, vol. 13 (1906), pp. 179–203.

of the problem of three bodies. It is the object of Professor Lovett's second note to write out the corresponding solutions for those cases of the problem of four bodies which consist of three finite bodies in motion according to one of the lagrangian exact solutions of the problem of three bodies and a fourth body infinitesimally small.

5. The paper of Professors Veblen and Young contains a completely independent* set of assumptions for projective geometry, in which points and undefined classes of points called lines have been taken as the undefined elements. The assumptions are so arranged that a certain group of eight characterize what may be called general projective spaces, *i. e.*, spaces in which the points can be represented by homogeneous coordinates which are elements of a finite or infinite number system, in which the operation of multiplication may or may not be commutative. On adding to this group a ninth assumption (from which the so-called fundamental theorem of projective geometry can be derived), we obtain a set of assumptions which characterize what we call properly projective spaces, namely spaces in which the points can be represented by coordinates which are elements of a commutative number system, *i. e.*, of a finite or infinite field. Spaces in which the coordinates are elements of modular and non-modular fields are then distinguished by a further assumption. Finally it is shown how by adding other assumptions one may arrive at categorical and completely independent sets of assumptions for the projective geometries in which the coordinates are on the one hand the ordinary real numbers, and on the other hand, the ordinary complex numbers.

6. Professor Morley's paper is in abstract as follows: Clifford's theorem on p lines of a plane (Works, page 38) gives a regular configuration T_p of 2^{p-1} points and as many circles; on each point are p circles and on each circle are p points. Bifocal curves of class n, on $3n - 3$ given lines, set up a Cremona transformation between the two real foci. This gives in certain cases a transformation of the points of a Clifford configuration into themselves. Thus for T_6 the transformation is

$$xy = m + \mu t, \quad \bar{x} + \bar{y} + \bar{a} = t(x + y + a).$$

The paper will be offered to the *Transactions*.

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* Ordinally independent sets have been given before; but so far as the authors are aware this is the first completely independent set.

7. In a previous paper, Professor Richardson has shown that the existence of the double and iterated integrals of any function is a sufficient condition for their equality. The present discussion covers the case where one only of the integrals is assumed to exist. By means of the notion of measure of an aggregate introduced by Borel, Lebesgue has defined * an integral which includes the ordinary (Riemann) integral as a special case. As far as is known, this integral exists for every limited function. By means of a generalization of the Lebesgue integral to the case of an unlimited function, it is shown that the existence of the double integral of any function $f(x, y)$ in a two-dimensional domain T is a sufficient condition for the equality

$$\int_T f(x, y)dT = L_Y dy \overline{\underline{\int}}_X f(x, y)dx,$$

where the symbol L denotes the Lebesgue integral. The existence of the integrals $\int_Y \overline{\underline{\int}}_X f$, $\int_Y \overline{\underline{\int}}_X |f|$ is a sufficient condition for the existence of the Lebesgue double integral and for the equality

$$\int_Y \overline{\underline{\int}}_X = L_T f.$$

Various other relations of a more general nature are derived and the results extended to the case where the domain T is unlimited.

8. Methods for partially computing the motion of the moon relative to a moving plane of reference have been given by Adams, Hill, Radau, and others. It is shown in Professor Brown's first paper that the general formulas for referring the motion to moving axes give a simple disturbing function for the general case where the periodic as well as the secular terms in the motion of the ecliptic are included.

9. In Professor Brown's second note the methods given by Hill (*Acta Mathematica,* volume 8) are replaced by the well-known formula for the development of a determinant in powers

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* Leçons sur l'intégration et la recherche des fonctions primitives, Paris, Gauthier-Villars, 1904.

of the constituents of the leading diagonal. The summing of the resulting series is made by a method which avoids the use of conditionally convergent series.

10. In order that a set of singular n-ary matrices (linear substitutions with vanishing determinants) may form a group in the ordinary sense (not in the broader sense in which the word is used by Frobenius and Schur in the *Berliner Sitzungsberichte* for 1906, I, page 209), all its matrices must have the same rank r, and their characteristic equations must have the same number of zero roots, viz., $n - r$. Every such group is reducible; in its reduced form every matrix consists of an ordinary r-ary matrix bordered with zeros. A singular matrix S will belong to some group if, and only if, the invariant factors of its characteristic equation corresponding to zero roots are linear. S belongs to a group of finite order if, and only if, the non-zero roots of its characteristic equation are roots of unity and the invariant factors of its characteristic determinant are all linear. S is the identity matrix of some group (idempotent) if, and only if, the roots of its characteristic equation are all equal to 0 or 1 and the invariant factors of its characteristic determinant are linear. Every matrix of rank n or $n - 1$ belongs to some group.

11. In a paper published in the *American Journal of Mathematics*, volume 24, Professor Hardy derived for curves in a space of four dimensions formulas corresponding to the Serret-Frenet formulas for curves in a three-dimensional space. The present paper has for its object the derivation of Serret-Frenet formulas for a curve of $n - 1$ curvatures in a space of n dimensions. The method employed is the kinematic method used by Darboux in his Théoric des surfaces, volume 1, chapter 1.

12. Except for the preliminary adjunction of the square root of the discriminant, Klein's "problem of the A's," which depends on two essential parameters, requires for its solution all the irrationalities and transcendents necessary in the solution of the general quintic. Dr. Coble shows that all the rational processes required to reduce the solution of the quintic to the problem of the A's can be set forth by invariantive methods. A form problem in which the known quantities are the values of the invariants is developed. The underlying G_{120} is a Cre-

mona group. The one necessary transformation of the quintic is accomplished by means of the linear covariants.

13. Given the differential equation $dy/dx = a(x, y)$, the problem of Dr. Hewes's paper is to determine the necessary and sufficient condition that this equation shall admit a one-parameter continuous conformal group. It is found that under the infinite group of all conformal transformations there exist, in connection with the differential equation above, invariant functions of A and its partial derivatives. There are two of the third order, from which two more of the fourth order are readily deduced. The necessary and sufficient condition is then that either of these invariants of the third order and the corresponding derived one of the fourth order are functions of the same harmonic function. If this condition is satisfied the group is uniquely determined.

14. Professor Hutchinson's paper appeared in full in the February BULLETIN.

15. The problems in the equilibrium of strings or chains which have been solved since the days when Bernoulli first investigated the catenary seem to be almost wholly those in which the field of force is rectilinear. One reason for this is probably the fact that fields other than the parallel and central fields lead to differential equations which are not linear. Professor Wilson treats the problem of equilibrium in a field compounded of gravity and a centrifugal force perpendicular to a vertical axis. This problem is readily recognized in the case of a string suspended at one end and forced to rotate at a uniform rate about a vertical axis through the point of suspension. Although the differential equation is not linear, it is of such form that it may readily be discussed qualitatively. At points considerably removed from the lowest point of the chain an approximate differential equation of the Bessel type may be obtained, and the qualitative discussion shows that in case the lower end of the string is free the configuration in the neighborhood of that point is also represented (qualitatively) to a remarkable degree by the approximate equation. One curious consequence brought out by the investigation is the existence of a rapidly increasing set of numbers $\alpha, \beta, \gamma \cdots$ such that, if the length l of the string is less than α, the string cannot depart from the vertical; whereas

if $\alpha < l < \beta$, there is one position of equilibrium and only one, if $\beta < l < \gamma$, there are two and only two positions of equilibrium, and so on. The paper will appear in the *Annals of Mathematics*, volume 9 (1908).

16. In the paper on the theory of double products and strains in hyperspace Professor Wilson presents: first, an account of so much of the late Professor Gibbs's lectures on multiple algebra as is essential to the understanding of his theory of dyadics; and second, some applications to the theory of strains and collineations in relation to involutory strains and collineations carried out by means of Gibbs's algebra. The results of this second part of the paper may be summarized briefly as follows: The determination of the square roots of the idemfactor by means of the properties of the equation of least degree; the determination of the necessary and sufficient conditions that a dyadic be resolvable into the product of two square roots of the idemfactor, and the correlation of these results with the theory of reflections in connection with unimodular strains and collineations; the introduction of certain invariant dyadics associated with any given dyadic and their application; the result that, if the Hamilton-Cayley equation of a dyadic is of least degree, the dyadic is resolvable into the product of three square roots of the idemfactor, of which the first is of the simplest type; the determination of the necessary and sufficient conditions that any dyadic be resolvable into the product of three square roots of the idemfactor.

17. The paper of Professor Mason is concerned with the determination of real implicit functions by an equation $f(x, y)=0$ when the partial derivatives of f of order less than n vanish at the point in question. Existence theorems are obtained with the aid of Dini's theorem, after applying certain transformations similar to those used in resolving higher singularities of an algebraic curve.

18. From the properties of matrices considered as linear vector operators, Professor Hawkes proved Weierstrass's theorems, that the necessary and sufficient condition for the equivalence of two non-singular families of bilinear forms is that their elementary divisors are equivalent.

19. For a partial differential equation of the first order

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0,$$

in which the first member is an analytic function of its five arguments, there exists in general through an analytic curve an analytic surface $z = z(x, y)$ which satisfies the equation. In the paper of Professor Bliss it is shown that a similar theorem is true when the function F is required to have continuous first and second derivatives only. The proof is made with the help of the theory of characteristic curves and the existence theorems for a set of ordinary differential equations

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \cdots, x_n), \cdots, \frac{dx_n}{dt} = f_n(x_1, x_2, \cdots, x_n).$$

The relation between the characteristic curves and integral surfaces is well known when everything is analytic. But in order to work the other way and derive existence theorems from the properties of characteristic curves, it is necessary to use the theorems on the differentiability of solutions of a system of ordinary equations with respect to the constants of integration. These theorems seem to have been overlooked in this connection.

F. N. COLE,
Secretary.

THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE twenty-second regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Monday, Tuesday and Wednesday, December 30–31, 1907, and January 1st, 1908, in connection with the fifty-eighth convocation of the American association for the advancement of science. The great gathering of scientists in other lines doubtless attracted unusual numbers of mathematicians, resulting in a wide representation of members and the largest attendance ever recorded at any meeting of the Society.

Monday afternoon and Tuesday morning were devoted to meetings with Sections A and D of the American association, for discussion of the teaching of mathematics to engineering students. These meetings proved of so great interest that an additional half-day was set apart for the informal discussion, and this session was finally brought to a most delightful climax by the vice-presidential address of Professor Edward Kasner of Section A, on "Geometry and mechanics," which proved equally fascinating to both engineers and mathematicians. A detailed report of these joint meetings, with abstracts of the papers, appears elsewhere in the BULLETIN, and the papers will be published in full in *Science* during the year.

The sessions on Monday morning, Wednesday morning, and Wednesday afternoon were given to the reading of papers. Professor Heinrich Maschke, Vice-President of the Society, presided at the first session, and Professor E. B. Van Vleck, Chairman of the Section, relieved at times by Professor L. E. Dickson and Professor H. B. Newson, presided at the other sessions.

The officers of the Section elected for the ensuing year were: Professor G. A. Miller, Chairman; Professor H. E. Slaught, Secretary, and Professor D. R. Curtiss, third member of the programme committee.

The promotion of acquaintance and good fellowship was an unusually prominent feature of the meetings, owing to the complete arrangements for entertainment and especially to the subscription dinner at the Hotel Del Prado, where just one hundred mathematicians and engineers were gathered. It has long been the custom on these occasions to send postal card greetings to absent members — a custom inaugurated by Professor E. H. Moore — and in this instance the hundred greetings were sent to him, on board ship departing for his year's vacation.

The attendance at the several sessions included the following eighty-four members of the Society:

Professor D. P. Bartlett, Professor S. M. Barton, Professor W. W. Beman, Dr. G. D. Birkhoff, Professor O. Bolza, Professor W. C. Brenke, Professor W. E. Brooke, Mr. Thomas Buck, Professor W. H. Bussey, Professor D. F. Campbell, Professor H. E. Cobb, Mr. E. H. Comstock, Professor C. E. Comstock, Professor D. R. Curtiss, Dr. H. N. Davis, Professor E. W. Davis, Professor S. C. Davisson, Professor L. E. Dickson, Professor J. F. Downey, Professor L. W. Dowling, Mr. Arnold Dresden, Professor H. T. Eddy, Professor C. C. Engberg, Mr. E. B. Escott, Professor H. B. Evans, Professor T. M. Focke, Mr. R. M. Ginnings, Miss Harriet E. Glazier, Professor B. F. Groat, Professor A. G. Hall, Professor G. B. Halsted, Professor Harris Hancock, Dr. Charles Haseman, Professor C. N. Haskins, Professor A. S. Hathaway, Professor E. R. Hedrick, Mr. T. H. Hildebrandt, Mr. F. H. Hodge, President C. S. Howe, Professor E. V. Huntington, Professor Edward Kasner, Professor O. D. Kellogg, Professor Kurt Laves, Dr. A. C. Lunn, Mr. W. A. Luby, Professor C. R. Mann, Mr. W. D. MacMillan, Professor J. L. Markley, Professor Heinrich Maschke, Professor G. A. Miller, Professor F. R. Moulton, Professor G. W. Myers, Dr. L. I. Neikirk, Professor B. L. Newkirk, Professor H. B. Newson, Mr. E. W. Ponzer, Professor Charles Puryear, Professor N. C. Riggs, Professor D. A. Rothrock, Mr. A. R. Schweitzer, Miss Ida M. Schottenfels, Professor G. T. Sellew, Professor J. B. Shaw, Mr. R. L. Short, Dr. C. H. Sisam, Mr. C. G. Simpson, Professor E. B. Skinner, Professor H. E. Slaught, Professor C. S. Slichter, Professor S. E. Slocum, Mr. E. R. Smith, Professor A. W. Smith, Mr. E. H. Taylor, Professor E. J. Townsend, Dr. A. L. Underhill, Professor E. B. Van Vleck, Professor C. A. Waldo, Professor C. B. Williams, Professor D. T. Wilson, Professor E. J. Wilczynski, Professor F. S. Woods, President R. S. Woodward, Mr. A. E. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

The following papers were read:

(1) Dr. J. C. MOREHEAD: "The rapid computation of power residues."

(2) Dr. C. H. SISAM: "On the inflectional tangents to a quartic curve."

(3) Dr. C. H. SISAM: "Some loci connected with plane curves, II."

(4) Professor J. B. SHAW: "Standard forms of certain types of Peirce algebras."

(5) Professor G. A. MILLER: "On the multiple holomorphs of a group."

(6) Professor L. W. DOWLING: "On the generation of plane quintics with ordinary double points."

(7) Professor E. J. WILCZYNSKI: "Projective differential geometry of curved surfaces, III."

(8) Dr. CHARLES HASEMAN: "Some boundary problems in the theory of functions."

(9) Dr. A. R. CRATHORNE: "Hilbert's invariant integral in the general isoperimetric problems."

(10) Mr. E. B. ESCOTT: "The converse of Fermat's theorem."

(11) Professor W. C. BRENKE: "Convergence, summability and differentiability of trigonometric series."

(12) Mr. A. W. SCHWEITZER: "On Moore's 'Tactical memoranda.'"

(13) Dr. G. D. BIRKHOFF: "On a certain existence and oscillation theorem."

(14) Professor E. H. MOORE: "Note on a form of general analysis."

(15) Professor L. E. DICKSON: "Reduction of families of quadratic forms in a general field."

(16) Professor L. E. DICKSON: "On commutative linear groups."

(17) Mr. H. E. BUCHANAN: "Note on the convergence of a sequence of functions of a certain type."

(18) Dr. A. L. UNDERHILL: "Note on the calculus of variations."

(19) Professor E. R. HEDRICK: "Note on the structure of point sets" (preliminary communication).

(20) Professor C. N. HASKINS: "Note on the generalized law of the mean."

(21) Professor F. R. MOULTON: "On certain relations, due to tidal friction, in the motion of two mutually attracting bodies."

(22) Professor A. S. HATHAWAY: "Motion of three bodies with fixed center of gravity."

Mr. Buchanan was introduced by Professor Moulton. In the absence of the authors the papers of Dr. Morehead, Dr. Crathorne, and Professor Moore were read by title. Abstracts of the papers follow below; the numbering is the same as that attached to the titles in the above list.

1. Dr. Morehead's paper describes a means of solving easily the congruence $a^{kn}x \equiv b \pmod{m}$ for successive values of k. The solutions x are nth power residues or non-residues, modulo m, according as b is an nth power residue or non-residue. The power residues, modulo m, of all degrees may be obtained by the solution of $\phi(m)$ (the totient function) congruences of the above form. The case $a = 2$ is the most important, because of its simplicity.

2. In this paper Dr. Sisam proves that the inflectional tangents to an arbitrary quartic curve form the complete set of tangents common to a curve of class four and one of class six. The curve of class four in question is the envelope of the lines meeting the quartic in equianharmonic points; the one of class six is the envelope of the lines meeting the quartic in harmonic points.

3. In a paper previously presented* Dr. Sisam discussed the locus of the points of intersection of the tangents to a curve when the lines joining their points of tangency touch a second given curve. In the present paper, Plücker's numbers for these loci are completely determined, several important special cases are considered, and the dual loci are discussed.

4. The following types of Peirce algebras (that is, nonquaternion algebras with no skew units) have previously been considered and the subtypes exhibited:

$(i, i^2, \cdots, i^{r-1})$, deficiency zero, by Benjamin Peirce.
$(i, j^2, \cdots, j^{r-2})$, deficiency unity, by Benjamin Peirce.
$(i, i^2, j, \cdots, j^{r-3})$, deficiency two, by Starkweather.
$(i, j, ij, j^2, \cdots, j^{r-3})$, deficiency two, by Starkweather.
$(i, j, k, k^2, \cdots, k^{r-3})$, deficiency two, by Starkweather.

It is the purpose of Professor Shaw's paper to consider further types and reduce the same to their simplest forms; in particular the types $(i, i^2, \cdots, i^m, j, \cdots, j^{r-m-1})$, $(i, j, \cdots, j^{m-1}, k, \cdots, k^{r-m-1})$, $(i, i^2, \cdots, i^m, j, ij, \cdots, j^{r-m-2})$, $(i, i^2, \cdots, i^m, j, ij, j^2, ij^2, j^3, \cdots, j^{r-m-3})$.

5. In considering the holomorph of an abelian group G of odd order, Burnside proved a theorem under the assumption that G is a characteristic subgroup of its holomorph (Theory of groups of finite order, page 238). Professor Miller proves that this assumption is superfluous, by establishing the theorem that an abelian group of odd order is always a characteristic subgroup of its holomorph. He considers the general question, under what condition an abelian group is a characteristic subgroup of its holomorph. Some of his results are expressed in the following theorems: If the order of an abelian group is not divisible by 8, it is a characteristic subgroup of its holomorph. If an abelian group K of order 2^m contains only one largest in-

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* BULLETIN, vol. 13, p. 277.

variant exceeding 2^3 and if its holomorph contains another invariant subgroup K_1 which is similar to K, then K, K_1 will generate a group whose commutator subgroup is of order 2. When K contains only one second largest invariant, its holomorph contains exactly three other invariant subgroups which are similar to K; but its holomorph contains only one other invariant subgroup similar to K when K contains more than one second largest invariant. The holomorph of any abelian group K cannot involve more than four invariant subgroups which are similar to K.

6. Castelnuovo (*Mathematische Annalen*, volume 44), has proved that every plane involution is rational, *i. e.*, that every such involution may be set up by a net of curves. Given, therefore, any involution I_n of order n, it is possible to find a net of curves such that any two curves of the net intersect in a set of points of the involution, and any two sets of points determine a curve of the net. The plane may be considered as containing two sorts of elements, the point sets of I_n and the curves of the net. The geometry in such a plane includes the discussion of all plane curves which can be the loci of singly infinite series of point sets of I_n or the envelopes of singly infinite series of curves of the net.

Professor Dowling considers the I_{6-d} set up by a net of cubics having three collinear points in common and d other points, where $d \leqq 4$. Quintic curves possessing simple double points appear among the involution curves. The converse problem, whether a given quintic, can be considered as an involution curve in an I_{6-d}, is answered in the affirmative.

Other involutions are considered with a view to applying the notion of an involution in the plane to the generation of any algebraic plane curve.

7. In a previous paper Professor Wilczynski has shown that the equation of every non-ruled surface can be developed, in the vicinity of an ordinary point, in the canonical form

$$(1) \qquad z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \cdots,$$

where I, J and all further coefficients are absolute invariants of the surface. No development of this form exists for a ruled surface. It is the purpose of the present paper to find a canonical development for this exceptional case and to discuss it in detail. The canonical form found is

$$(2)\quad z = xy + x^3 + x^5 + Ax^4y + Bx^3y^2 + \text{terms of the sixth order},$$

where A and B are absolute invariants. Certain exceptional cases are also discussed in which this new development also breaks down.

The question as to the meaning of the canonical development (2), *i. e.*, primarily as to the geometric significance of the tetrahedron of reference which gives rise to this development, is then taken up and completely answered. A number of new and important results are found on the way. It is found that there exists a single infinity of cubic scrolls which have contact of the fourth order with the given ruled surface at the given point. Each of these scrolls has a nodal straight line with two pinch points upon it. The locus of the nodal lines is the osculating hyperboloid. The locus of the pinch points is a twisted cubic curve upon the osculating hyperboloid.

Among this single infinity of cubic scrolls there is one whose pinch points coincide (a Cayley cubic scroll). Thus there exists one and only one Cayley cubic scroll which has contact of the fourth order with a given ruled surface at a given point. The pinch point, nodal line, and singular tangent plane of this surface are respectively a vertex, an edge, and a face of the canonical tetrahedron. Thereby the tetrahedron is completely determined. For the two edges which meet at the given point of the surface are the two asymptotic tangents of the surface at that point.

There is thus a unique Cayley cubic scroll associated with every point of a ruled surface. The question arises as to the relation between those which belong to points of the same generator. It is found that this relation is as follows: the locus of the pinch points of the osculating Cayley cubic scrolls, which belong to the various points of any generator, is a twisted cubic curve which is situated on the osculating hyperboloid, and which intersects the generator in its flecnodes.

8. In this paper Dr. Haseman first shows how to prove the existence of a regular analytic function of a complex variable $f(z) = u(x, y) + iv(x, y)$ on a multiply connected as well as on a simply connected surface, where the relation

$$a(s)\,u(s) + b(s)\,v(s) + c(s) = 0$$

between the boundary values $u(s)$ and $v(s)$ of the real and imaginary parts respectively of $f(z)$ is given, and where $a(s)$, $b(s)$, $c(s)$ may each have a finite number of finite discontinuities. This is done by choosing a regular analytic function $g(z)$ in the given region such that $f^*(z) = f(z) g(z)$, where $f^*(z)$ may be gotten from the relation $a^*(s) u^*(s) + b^*(s) v^*(s) + c^*(s) = 0$, and where the coefficients in the relation are all continuous throughout.

A number of special cases arise from this theory. For example, a regular analytic function $f(z)$ may be found, where the real part $u(s)$ is given for certain segments of the bounding curve of the given region and the imaginary part $v(s)$ for the rest of the boundary.

A similar problem may be solved for multiply connected surfaces; for example, we may find $f(z)$ for the region between two concentric circles, where $u(s)$ is given on the outer circle and $v(s)$ on the inner circle.

It is next shown how to find a regular analytic function of a complex argument $f_a(z)$ exterior to a given region and another function of the same character $f_j(z)$ interior to the region, when the relation $f_a[\sigma(s)] = c(s) f_j(s)$ is given between the boundary values $f_a(\sigma)$ and $f_j(s)$ of our functions at the points $\sigma = \sigma(s)$ and s respectively, where $\sigma(s)$ is a regular differentiable function of s and where $c(s)$ may have a finite number of finite discontinuities.

The required functions are found by choosing two other functions $g_a(z)$, $g_j(z)$ of the same characters such that $f_a^*(z) = f_a(z) g_a(z)$ and $f_j^*(z) = f_j(z) g_j(z)$, in which the starred functions are gotten from the relation $f_a^*[\sigma(s)] = c^*(s) f_j^*(s)$, where $c^*(s)$ is everywhere continuous. The theory may be extended to multiply connected surfaces.

Finally the existence is proved of two regular analytic complex functions $f_a(z)$, $f_a'(z)$ exterior to a given region and two others of the same character $f_j(z)$, $f_j'(z)$ inside the given region, where the following relations

$$f_a(\sigma) = c_1(s) f_j(\rho) + c_2(s) f_j'(s), \quad f_a'(\sigma) = c_1'(s) f_j(\rho) + c_2'(s) f_j'(s)$$

are given between the boundary values of $f_a(z)$ and $f_a'(z)$ at the point $\sigma = \sigma(s)$, of $f_j(z)$ at the point $\rho = \rho(s)$ and of $f_j'(z)$ at the point s of the boundary, where σ is a regular differentiable function of ρ, and ρ in turn the same kind of a function of s. The method of determining these functions is to build up two integral equations in f_j and f_j' similar to those found by Hilbert (*Göttinger Nachrichten*, 1905) for a somewhat simpler problem.

9. Dr. Crathorne considers the problem of determining the minimum of

$$J \equiv \int_1^2 F(y_1', \cdots, y_n'; y_1, \cdots, y_n; x)dx = \text{minimum}$$

under the conditions

$$J_i \equiv \int_1^2 G_i(y_1', \cdots, y_n'; y_1, \cdots, y_n; x)dx = l_i \quad (i = 1, 2, \cdots, r).$$

From the $(2n + r)$-parameter set of solutions of the Euler equations of the problem, it is possible to choose an n parameter set satisfying certain conditions analogous to the condition of transversality in the plane problem. The integral

$$J^* \equiv \int_1^x \{F^* + \sum F^*_{P_i}(y_i' - P_i)\}dx \quad (i = 1, 2, \cdots, n),$$

where $F^* = F + \sum \lambda_i G_i$, is the solution of the Hamilton equation of the problem. Each solution of this equation determines a set $P_1, P_2, \cdots, P_n$ such that the solutions of $y_i' = P_i$ form an n-parameter set which is shown to be equivalent to the above n-parameter set of solutions of Euler's equations. A "field" is thus established in which J^* is a point function.

10. The paper by Mr. Escott discusses the converse of the theorem of Fermat. If p is any prime and e any integer not divisible by p, $e^{p-1} - 1 \equiv 0 \pmod{p}$. According to Ball's Mathematical recreations, the Chinese have a theorem that the condition $2^{p-1} - 1 \equiv 0 \pmod{p}$ is a sufficient condition for a prime number. This theorem has been shown to be untrue by a number of persons. In this paper, which is based on one published in the *Messenger of Mathematics* in March, 1907, Mr. Escott gives a method by which composite numbers can be found containing as many factors as desired and which satisfy the above congruence.

11. In this paper Professor Brenke considers the series

$$\sum_1^\infty (a_u \cos ux + b_u \sin ux),$$

and determines sufficient conditions for convergence, summability (based on the definition of Cesàro) and existence of derivatives of the function defined by the series. The results are obtained by transformation by a factor

$$\phi(x) = R(\cos \tfrac{1}{2}bx, \sin \tfrac{1}{2}bx),$$

R denoting a rational integral function of the arguments indicated. Application is made to increasing the rate of convergence of trigonometric and other series, and the sums of certain classes of series are obtained in closed form.

12. In his "Tactical memoranda"* Moore defines systems of k-ads, k-ids, $\underset{\cdot}{k}$-ads, $\underset{\cdot}{k}$-ids. A k-ad may be thought of as any one of a symmetric class of k-ids in k elements and hence may be called a symmetric k-id. A $\underset{\cdot}{k}$-id is some one of the sets of 2-ids obtained from the open linear chain $x_1x_2 \quad x_2x_3 \cdots x_{k-1}x_k$ of 2-ids in k distinct variables, each variable having the range $a_1a_2 \cdots a_k$ independently. Thus a $\underset{\cdot}{k}$-id is what Mr. Schweitzer elsewhere has called an ordered k-ad. A k-id is simply an open linear chain of 2-ids in k elements. A closed linear chain of 2-ids in k elements is a Moore $S\{2, 1k\}$ but not necessarily conversely.

Between the proper k-idic, and the symmetric k-idic is the alternating k-idic standpoint; the latter is considered chiefly in Mr. Schweitzer's paper. A set of connected k-ids with the index of connection (k, p), $p = 1, 2, \cdots, k$ is defined; every k-id is a set of connected k-ids, index (k, k). For example, the 3-ids *mbc*, *amc*, *abm*, have the index $(3, 1)$; the 3-ids *mbn*, *mnc*, *amn*, have the index $(3, 2)$, etc. Corresponding to each type of connection, types of open and closed chains of k-ids are defined. A set of alternating k-ids with index of connection $(k, 1)$ is reduced formally to chains with index $(2, 1)$, and these chains are also formed by the suitable superposition of linear chains of 2-ids. Mr. Schweitzer has elsewhere defined an n-dimensional permutation $(n = 1, 2, 3, \cdots)$ and shown that every such permutation contains an n-dimensional open chain. It is here shown that every n-dimensional chain is reducible to chains with index $(2, 1)$. Finally, the resolution of an alternating set of k-ids in m elements into chains of alternating k-ids is discussed. This leads to infinitudes of finite systems which are important for independence considerations in the author's geometric systems. Particular cases of Moore's $S\{k, l, m\}$ arise frequently and in an interesting synthetic manner.

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* *Amer. Jour. of Math.*, vol. 18, pp. 264–303. The possibility of a connection between the tactical systems here set forth and the above memoir was suggested by Professor Moore. The author takes great pleasure in acknowledging the stimulating influence of Professor Moore and the unusual advantages which have thus been most generously placed at his disposal.

13. Dr. Birkhoff's paper develops an existence and oscillation theorem for the solutions of a differential equation

$$\frac{d^2u}{dx^2} + q(x, \lambda)u = 0$$

fulfilling self-adjoint boundary conditions at a and b. The function $q(x, \lambda)$ increases with the parameter λ. The methods are similar to those which Professor Bôcher has employed to consider a special case. The paper will be offered to the *Transactions.*

14. The note of Professor Moore will appear in full in an early number of the BULLETIN. The form of analysis in question is that in which one at least of the independent variables enters without direct specification of nature and range of variation.

15. The first paper by Professor Dickson is devoted, in the main, to the reduction, to non-equivalent types, of families of ternary quadratic forms based on two linearly independent forms. The analysis applies to a general field or domain of rationality. The paper has been offered for publication in the *Quarterly Journal of Mathematics.*

16. The paper by Professor Dickson on commutative linear groups is a continuation of the one presented at the last summer meeting (abstract in BULLETIN, November, 1907, pages 61, 62). The complete article has been offered for publication in the *Annals of Mathematics.*

17. In this note Mr. Buchanan states and proves the following theorems: (1) If a sequence of functions, monotonic non-decreasing, converges to a continuous limit function, then the limit function is monotonic non-decreasing and the convergence is uniform. (2) If a sequence of functions $f_n(x)$ $(n = 1, 2, 3, \cdots, n)$ be of finite variation in an interval ab and

convergent to a continuous limit function $f(x)$, also of finite variation in ab, and if at every two points x_1, x_2, $a \leqq x_1 < x_2 \leqq b$, the limit of the variation of $f_n(x)$ is equal to the variation of $f(x)$, then the convergence is uniform in ab.

18. In a paper presented to the Society in April, 1907, Dr. Underhill showed an invariantive normal form of the second variation of

$$J = \int_{t_0}^{t_1} F(x, y, x', y')dt,$$

namely,

$$\delta^2 J = \int_{\alpha_0}^{\alpha_1} \left[\left(\frac{d\nu}{d\alpha}\right)^2 - \bar{\kappa}_0 \nu^2 \right] d\alpha, \text{ where } \alpha = \int_{t_0}^{t} F(x, y, x', y')dt$$

and where ν and $\bar{\kappa}_0$ are absolute invariants under extended point and parameter transformations connected with the integrand function of J.

The Jacobi equation in this case is

$$\psi(\nu) \equiv -\bar{\kappa}_0 \nu - \frac{d^2\nu}{d\alpha^2} = 0.$$

Let $\nu = \phi(\nu)$ be a solution vanishing for $\alpha = \alpha_0$. In order to draw conclusions as to the next zero of $\nu = \varphi(\alpha)$, use is made of the equation

$$\frac{d^2\nu}{d\alpha^2} + \frac{1}{a^2}\nu = 0,$$

which has as a solution vanishing at $\alpha = \alpha_0$

$$\nu = c \sin\left(\frac{\alpha - \alpha_0}{a}\right),$$

and with the next zero at $\alpha = \alpha_0 + \pi a$.

By means of two theorems on differential equations (cf. Darboux, Théorie des surfaces, III, § 629), by which the solutions of the Jacobi equation are compared with those of the last equation, the following theorems may be at once formulated :

I. For the problem of minimizing J, if $1/\bar{\kappa}_0$ is positive and less than a^2 along the extremal, then the extremal cannot be a minimizing curve in an interval greater than πa.

II. If $1/\bar{\kappa}_0$ is positive and greater than b^2 along the extremal, then the extremal furnishes a weak minimum in an interval at least equal to πb.

III. In case the problem is that of geodesics on a surface, $\bar{\kappa}_0$ is the gaussian curvature and the above theorems reduce to those already stated by Bonnet, *Comptes rendus*, volume 40, page 1311, also volume 40, page 32.

19. In this preliminary paper Professor Hedrick discusses certain consequences of the Lindelöf definition of a point of condensation (see Borel, Leçons, 1905, Chapter I). Many of the results are not new and a detailed statement is withheld pending further investigation along the lines indicated.

20. The paper of Professor Haskins is a geometric interpretation of L' Hôpital's theorem on the generalized law of the mean. It will be offered to the *Annals of Mathematics* for publication.

21. The motion of a system of tidally disturbing bodies is conditioned by the conservation of the moment of momentum with respect to any plane, and by the fact that the total kinetic and potential energy of the system can only decrease. Professor Moulton's paper develops the consequences of these conditions in the various cases which can arise. Applications are made chiefly to the earth-moon system, and to double stars. Among other things it is shown that tidal friction can not have been an important factor in developing the distances between the stars in binary systems.

22. Professor Hathaway considers a reduced problem for arbitrary motion, analogous to Lagrange's reduced problem for gravitational motion. The reduced elements are the three mutual distances and three relative velocities of the bodies (in magnitude only). When these are given, we can determine algebraically the angle between the plane of the bodies and the plane parallel to their relative velocities, the angular velocities of these planes, their inclinations to the axis of moment of momentum, etc. The complete solution of the problem is by linear differential equations in terms of the reduced elements whose complete solutions are arbitrary constant rotations of a particular solution. When one set of these differential equations is

solved, the other set may be reduced to quadratures. If the axis of moment of momentum is fixed, the complete solution is by quadrature, thus paralleling the gravitational problem.

It also appears that Lagrange's investigations in the problem of three bodies are generally independent of the law of gravitation. For example, those motions in which the triangle of the bodies is always similar to itself require that triangle to be either equilateral or with collinear vertices, and the sides to revolve in a fixed plane through the center of gravity. It is shown here that the angular velocity of the triangle varies inversely as the square of a side and so is constant for rigid configurations. Also that no other solution than the above is possible for constantly collinear bodies whose co-line is not a fixed line.

Lagrange's biquadratic equation for ρ holds also for arbitrary motion, and his differential equation for the same in any gravitational motion has the roots of that biquadratic for particular, and not singular, solutions.

H. E. Slaught,
Secretary of the Section.

JOINT MEETINGS OF MATHEMATICIANS AND ENGINEERS AT THE UNIVERSITY OF CHICAGO.

A series of meetings of mathematicians and engineers was held at the University of Chicago, December 30–31, 1907, under the auspices of the Chicago Section of the American Mathematical Society and conjointly with Sections A (mathematics and astronomy) and D (mechanical science and engineering) of the American Association for the Advancement of Science.

The invitation to join in the discussion of the teaching of mathematics to students of engineering had been widely distributed among those engaged in the practice of engineering as well as among professors in technical schools. The attendance was large and representative, including one hundred men especially interested on the mathematical side and fifty on the engineering side. Among the institutions represented were the State Universities of Illinois, Indiana, Iowa, Kansas, Minnesota, Missouri, Nebraska, Ohio, Pennsylvania, Vermont, West Virginia and Wisconsin; the following technical schools: Armour Institute of Technology, Bradley Polytechnic Institute, Case School of Applied Science, Lewis Institute of Chicago, Massachusetts Institute of Technology, Michigan College of Mines, Michigan Agricultural College, Purdue University, Rose Polytechnic Institute, Stevens Institute of Technology, and the Agricultural and Mechanical College of Texas; and the following institutions: Brown University, The Carnegie Institution of Washington, University of Chicago, University of Cincinnati, Colgate University, Harvard University, Knox College, Miami University, James Millikin University, Kalamazoo College, New Hampshire College, Oberlin College, Ohio Wesleyan University, University of the South, Union College, Washington University, Wheaton College, and the United States Weather Bureau.

The promotion of acquaintance and good fellowship was an important feature of the meetings, which was fostered by the admirable arrangements provided by the University of Chicago and especially by the subscription dinner attended by one hundred engineers and mathematicians. The speakers at the dinner, introduced by Professor E. B. Van Vleck, Chairman of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY, were Professor Calvin M. Woodward, Dean of the School of Engineering of Washington University, St. Louis, Mo., Mr. Charles F. Scott, Chief Consulting Engineer of the Westinghouse Electric and Manufacturing Company, Pittsburgh, Pa., Professor George A. Swain, Massachusetts Institute of Technology, Boston, Mass., and Professor Edward V. Huntington, Harvard University, Cambridge, Mass.

The presentation of the papers, covering two half days, led to an enthusiastic discussion which extended to a third session and which culminated in the appointment of a committee to take into consideration the whole question of the mathematical curriculum in technical schools and in technical departments of colleges and universities, and to report to a joint meeting of engineers and mathematicians to be held in 1909 in connection with the annual gathering of the Society for the promotion of engineering education. The selection of this important committee was entrusted to Professor E. V. Huntington of Harvard University, Professor Gardner S. Williams of the University of Michigan, and Professor E. J. Townsend of the University of Illinois, who themselves are to constitute the nucleus of the committee and who were given full power to select the remaining members, choose a chairman, and determine the scope of the investigation to be entered upon, it being understood that the committee shall number not less than fifteen and shall be widely representative both geographically and as to all the institutions and interests involved.

At the first session four papers were presented as follows :

"The present status of mathematical instruction for engineers in American schools," Professor E. J. TOWNSEND, University of Illinois.

"The present status of mathematical instruction for engineers in foreign schools," Professor ALEXANDER ZIWET, University of Michigan.

"The point of view in teaching engineering mathematics," President R. S. WOODWARD, The Carnegie Institution of Washington.

"The scope and spirit of mathematical instruction for engineers," Mr. CHARLES F. SCOTT, consulting engineer, Westinghouse Electric and Manufacturing Company.

The second session consisted of a symposium on the topic : "What is needed in the teaching of mathematics to students of engineering?" The subdivisions of the topic were :

(*a*) What range of subjects? (*b*) To what extent in the various subjects ? (*c*) By what methods of presentation ? (*d*) What should be the chief aims?

The speakers, who represented three phases of the subject, namely, (1) the standpoint of the practicing engineer, (2) the standpoint of the professor of engineering, (3) the standpoint of the professor of mathematics in the engineering school, were as follows :

RALPH MODJESKI, consulting civil engineer, Chicago, Ill.

CHARLES S. SLICHTER, professor of applied mathematics and consulting engineer, University of Wisconsin.

GARDNER S. WILLIAMS, professor of civil hydraulic and sanitary engineering and consulting engineer, University of Michigan.

FREDERICK S. WOODS, professor of mathematics, Massachusetts Institute of Technology, Boston, Mass.

GEORGE F. SWAIN, professor of civil engineering, Massachusetts Institute of Technology, Boston, Mass.*

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* Professor Swain, on short notice, supplied the place of Mr. W. L. Abbott, chief operating engineer of the Chicago Edison Company, who was unexpectedly unable to be present.

ARTHUR N. TALBOT, professor of municipal and sanitary engineering in charge of theoretical and applied mechanics, University of Illinois.

FRED. W. MCNAIR, president of the Michigan College of Mines, Houghton, Mich.

J. A. L. WADDELL, consulting bridge engineer, Kansas City, Mo., whose paper was read by Professor O. D. Kellogg.

The papers will be printed in full during the year 1908 in *Science* and will thus reach all who are members of the American association for the advancement of science.

Abstracts of the papers and a list of those who contributed to the general discussion are given below:

Professor Townsend considers the rapidly increasing demand for trained men in the field of engineering and the remarkable growth in recent years of the technical schools, both in number and in equipment and efficiency, and shows that the time is indeed ripe for serious consideration, on the part of both mathematicians and engineers, of all phases of the question of mathematical training for engineering students. He has selected for comparison seventeen institutions in this country where engineering training is either an important or the exclusive feature and he considers three questions with respect to the mathematical training of engineering students in these institutions, as representative of the status in the country at large; namely, (1) the entrance requirements, (2) the requirements for graduation, (3) the qualifications of the instructional force. Under the first and second of these heads Professor Townsend exhibits a comparative tabular statement and draws a number of incisive conclusions both as to the facts and as to the tendencies, all of which may well be pondered carefully by everyone interested in the industrial and scientific development in America. These data and preliminary observations will be of prime importance to the committee of fifteen who are to investigate and report upon the whole question of the mathematical curriculum in the technical schools. Under the third head also Professor Townsend has clearly set forth the difficulties and the demands of the situation, and it will be the province of the general committee to report also upon these matters. It is greatly to be desired that all mathematicians may not only read this paper, as it will appear in full in *Science*, but that many suggestions may be communicated to the committee through Professor Townsend. To this end the following questions suggested by his paper are proposed:

I. ENTRANCE REQUIREMENTS.

1. Is a greater uniformity of entrance requirements desirable?
2. Most colleges in the middle west admit on certificate from accredited schools. In addition to this should engineering students be required to pass an entrance examination in algebra, with the understanding that if they fail to make satisfactory grade, more than the usual amount of work must be done to secure credit in college algebra?
3. Should a knowledge of logarithms and the plotting of simple algebraic curves be added to the entrance requirements?
4. Should the standard of admission be raised so as to include trigonometry and college algebra?
5. Should the requirements be made to cover less ground and be correspondingly intensified?
6. Should more attention be paid to analytic and formal work, particularly in arithmetic and algebra?
7. Should a year of work in mathematics and science of college grade be required for entrance, or should the college course be extended to five years?

II. REQUIREMENTS FOR GRADUATION.

1. What should be the relative length of time spent on algebra, trigonometry, analytical geometry, and calculus?
2. Which should precede, algebra or trigonometry?
3. What topics should be particularly emphasized in college algebra? in calculus?
4. How far ought the instruction of the first two years' work in mathematics to be made "practical"; how far should we insist upon rigorous demonstrations of principles taught?
5. Should students in one line of engineering, say civil engineering, be given problems of a different nature than those given to students in other lines, say mechanical or electrical engineering?
6. Should differential equations and least squares be required subjects in any engineering course? If so, how extensive should these courses be?
7. Should we have a separate course on "applications" having for its purpose the cultivation of ability for rapid com-

putation and the use for engineering work of such instruments as the slide rule, planimeter, integraph, computing machines, etc.?

8. What opportunity for the study of pure mathematics should be given the engineering student beyond the usual course in calculus? What courses might be made elective in the junior or senior years?

9. Should a first course in mechanics be given to engineering students in the freshman year and before the student has had calculus?

III. ADMINISTRATIVE QUESTIONS.

1. What qualifications should we insist upon for the instructor of engineering students in mathematics?

2. How much elementary mechanics should be taught in connection with the calculus? Should this elementary mechanics be taught by the mathematical department?

3. Should the work in descriptive geometry be made more mathematical in treatment? Should it be taught by the mathematical department?

4. What can be done in general to bring about a closer relation between the teachers of mathematics and the teachers of engineering?

Professor Ziwet refers to the fact that remarkable and characteristic developments have taken place in England and Scotland, and that France represents probably the highest standard of any country with respect to mathematical training for engineering students, requiring, for instance, for admission to the Ecole polytechnique almost as much mathematics as would be required for graduation from our technical colleges. He considers, however, in some detail only the technical curricula of the German schools with which he is personally familiar, and reaches the general conclusion that while the average German engineer may ultimately know no more mathematics than the average American engineer, yet an able German student in his technische Hochschule, or engineering university, can get a more thorough scientific equipment than an equally able American student in his alma mater. One important advantage in the training of the German student is the fact that his preparatory mathematics (including arithmetic) is distributed systematically and continuously over a period of nine years, whereas in American schools there is much discontinuity and lack of homogeneity all along the line from the grammar school through the high school and freshman and sophomore years of the college course. Professor Ziwet gives an excellent resumé of the three phases of reform in the teaching of mathematics which are now commanding attention; namely, one which was originated by the German association of engineers, one which was promulgated by the university authorities, and a third which was proposed in connection with the committee of the German association of Naturforscher and Aertzte. From all these considerations Professor Ziwet draws thoughtful conclusions as to the situation in America and suggests some timely notes of warning.

President Woodward maintains that the fundamental ideas which belong to mathematics and mathematical physics and which form an essential part of engineering training are more inherently difficult than we commonly suppose, and that failure to recognize this in our teaching is the most glaring fault. We marvel at the ease with which the student forgets or fails to comprehend what we try to teach him. We must expect this and must therefore patiently continue to insist upon attention to the fundamental and elementary notions, and drive them home by unremitting practice in computational applications. President Woodward would have most of our text-books remodelled so as to present first the plain statement of facts and much later the theoretical and abstruse considerations. He takes pride in the high development in this country of pure mathematics and of experimental physics, but deplores the conspicuous lack of men highly trained in both mathematics and physics such as the French have produced — who combine both the theoretical and the practical and who therefore are prepared to advance the standards of scientific attainment in engineering lines. He attributes the dearth of such men in the theoretical field and in the teaching profession to the alluring remunerations held out to those who enter the commercial side of engineering work. President Woodward finds great hope for the future in this coöperation of mathematicians and engineers which has been inaugurated in these joint meetings and which bids fair to be perpetuated and extended through the mutually cordial sentiment here expressed and especially through the forces here put into operation.

Mr. Scott's paper presents many points for serious consideration to the teacher of pure mathematics. He appreciates fully the theoretical point of view with respect to the training of an engineer, yet he is in a position to realize how intensely practical the average engineer must be and how little use he has for any mathematics which he cannot instantly translate into the terms of his work-a-day environment. At the outset he discusses three questions: What are the uses to which an engineer may apply mathematics? What kind of mathematics does he need? What skill does he require in the use of mathematics? He then proceeds to discuss what mathematical subject matter should be covered by a student of engineering and how it should be taught. In this connection he lays great stress upon the training of the judgment and the development of the mathematical *sense*. It is one thing to know higher mathematics, or mathematics of any kind, and quite another thing to know how to use this knowledge in the practical world in bringing things to pass. "The ability to state a problem, to recognize the elements which enter into it, to see the whole problem, not overlooking some important factor, to use good judgment as to the reliability of the data involved, to be able to interpret the result, to recognize its physical significance — such powers as these are of higher order and of greater significance to the engineer than the ability to take a stated problem and work out the answer. It may be said that this is not strictly mathematics, but it is just the judgment and insight which make mathematics really useful and effective." Mr. Scott concludes with a paragraph on the relation of education to the affairs of life which every teacher of mathematics may well ponder.

Mr. Modjeski speaks from the standpoint both of a man highly trained in the Ecole des Ponts et Chaussées of France and also of a highly successful consulting bridge engineer of Chicago. While he would in no wise depreciate the stimulating effect of the mental discipline derived from a broad general course in the higher mathematics, he would call attention to the present agitation in France and Germany, especially in France, looking toward the curtailment of the current mathematical program in the engineering schools, on the ground that it is unnecessarily extensive, considerably more so than in this country. Mr. Modjeski would raise the question as to whether we should not specialize the mathematical courses and select for each branch of engineering those particularly adapted to training in that line. For instance, a railroad engineer who aspires to become a railroad official requires less knowledge of the calculus than the electrical or bridge engineer, but on the other hand he requires a greater knowledge of geology and common law than either of these. The bridge engineer demands a higher mathematical training than any other branch of the profession, and yet such a course as the theory of differential equations will be of little if any use to him and his time might better be given to such a course as, for instance, the methods of least work, which no bridge engineer should neglect. Mr. Modjeski would advocate the replacing of the abstract and meaningless problems of the elementary text-books by problems of a more practical and interesting nature, and above all he would insist upon a training in mathematical sense, a development of the habit of mathematical thinking to such a degree that the prospective engineer shall be able to apply the fundamental principles as one to whom they have become his second nature.

Professor Slichter points out that, whereas engineering technology was founded chiefly on practice rather than on theoretical principles, it has now grown to be an investigative science of such proportions that few scientific productions now excel the engineering and technological treatises which come from the press. The need of the engineer is not so much that his mathematics shall be taught to him in some different way as that he shall have *more* mathematics. The engineer does not ask for better instruments or more difficult projects or more capital to conduct operations, but for more knowledge. It is a hopeful sign that many engineering institutions are now fostering some phase of investigative work. The chief requirement of the mathematical curriculum is that it shall be compact and stable. There is very little time for the engineering student to dally in mathematics with whims and fads and new schemes. Mathematics is to the engineer not merely a tool, but like physics and chemistry, it is to him a basal science, and should be made to appeal to his interest and his experience. He will forget his mathematics, to be sure, but he will forget his hydraulics also, and the one as easily as the other, if they are not both intrenched by strong ties of interest. Professor Slichter would plead for more opportunity for electives in mathematics in engineering courses but would insist that, so far as the teaching is concerned, any shortcomings in the engineering profession are quite as likely due to faults in the technological courses themselves as in the mathematical courses.

Professor Williams recalls the fact that engineering a half century ago was merely an art, but that it has now risen to the standing of a science as well as an art, and that from this wider point of view only does mathematics have a place in the training of the engineer. Engineering as a science is hampered by the trade school which naturally views it as an art only. The products of the trade school have little use for the mathematics of engineering. Both the trade school and the highly specialized technical school have their place. While the vast majority of those connected with engineering work possess and use a very limited mathematical training, yet we may well consider the mathematical equipment demanded by the man who is to climb to the top of his profession. Professor Williams maintains that what the trained engineer needs is not more formulas but a better understanding of principles and a stronger development of the reasoning power, so that he may indeed be able to think mathematically. To this end the mathematical teaching of the grammar school needs overhauling and next that of the high school, the implication being that too much formal and abstract work is now required, and too little development of independent thinking through the solution and interpretation of practical problems. Mathematics is a tool but it is to be used with intelligence and not in the blind following of rules; it is, so to speak, a living instrument which must respond like an intelligent servant. As to the question of increasing the mathematical requirements for engineering students, there is danger that this may work a disadvantage if this is done at the sacrifice of physics, chemistry, and the other sciences. Elective courses in mathematics should be provided in the engineering course but not till the fourth year, as few students are competent to elect intelligently before that time.

Professor Woods refers to the fact that while the best technological schools offer ample elective work in such mathematical courses as advanced calculus, least squares, differential equations of physics and mechanics, etc., yet their chief concern is not the development of mathematicians as such, but the training of engineers who shall be able to use mathematics in both practical and theoretical work. He has no patience with the professors of engineering who undertake to direct the work of mathematics in an engineering school. He believes that the teachers of mathematics must be mathematicians and that, while they should constantly connect their work with the concrete side, yet they must teach mathematics as mathematics. He believes that the teaching of mathematics, like all other teaching, is capable of improvement and that great improvement has been made and is now in progress, but he is not willing to admit that the teaching in this line is worse, or has been worse, than in other lines, as some engineers would seem to imply. The mathematical department delivers its product to the engineering department of the technical school, there to be subjected to the searching critical tests of further years of training, while the engineering department delivers its product to the outside world there to be swallowed up in the multitude, whose criticisms are aimed at the man himself rather than at those who have been his teachers. Professor Woods gave a clear exposition of the fundamental steps needed in the successful teaching of mathematics in its relation to its concrete applications and elicited much interest in the modification of the curriculum now being put into effect at the Massachusetts Institute of Technology.

Professor Swain is doubtful as to the worth of mathematical study other than as a means to an end — the development of a tool with which to do something. He is disposed to assign low value to mathematics as mere training in mental power, in comparison with some other branches of science. He would criticize the present teaching of mathematics as too ideal and abstract, looking for logical and theoretical results which conform to the demands of rigid demonstration regardless of common sense, clear observation, and good judgment. From the standpoint of the engineer, for whom mathematics is simply a useful tool, and whose training, therefore, should lead to power in handling the instrument, Professor Swain would point out that the present unsatisfactory results may be due in part to the lack of consecutiveness in the mathematical curriculum, especially in the latter part of the secondary course, which permits a lapse of from one to three years in the study of algebra, and also at other points in the curriculum allows the presentation of subjects in too isolated and incoherent fashion. He also asserts that too much attention is given to pure analysis and not enough to geometry. To the engineer, he says, geometry is all important. The geometric demonstration is grasped by the mind and comprehended step by step, while analysis is like a machine which transforms the data into the desired conclusion by some occult process which the mind does not fully grasp. He would also revive the study of mental arithmetic and lay far more stress on all the mental processes of mathematical work. Finally he would discredit the lecture system of teaching mathematics, and would insist upon the Socratic method of question and cross-question in all class work, and would have the teaching of mathematics done by engineers or at least by men acquainted with the engineering applications of mathematics.

Professor Talbot discusses the mathematical training of engineering students from three points of view; namely, theory, practice, and philosophy or interpretation. His contention is that a proper relative proportion of emphasis on these three phases is of more importance than the content or extent of the curriculum, provided the principles covered are really understood and intelligently interpreted and applied. He emphasizes the fact that we must deal with the average student and that it is only the exceptional student who is the mathematical genius, hence the methods of presentation must be such as to encourage and develop the great middle class of engineering students. Professor Talbot believes that, while formerly we may have carried theoretical and demonstrational processes to an extreme, we are now likely to swing to the other extreme, and let the student conclude that the facts are of chief importance and the proof of principles involves useless effort. He would insist that practice in analysis and formal demonstration is illuminating and developing to the mind, that even the old time mental arithmetic had its great educational value. Likewise, practice and drill work have their uses and abuses. To the average student the working of problems is illuminating and educative. Mathematics is a tool and the engineering student must acquire facility in its use and this demands drill and repetition. But the students who think that to accept facts and work problems is sufficient, and the instructors who think that illustration and practice alone constitute mathematical training or that mere laboratory methods suffice, are greatly mistaken. Again, there must be a direct connection between the theory and the philosophy of the subject to make the practice side serve its proper purpose. Professor Talbot would discourage the lecture method for engineering students and would select with great care the more advanced and complex courses which are offered, especially those involving concepts lying beyond the student's experience.

President McNair considers not so much the ideals which we might set up for the mathematical curricula of the engineering schools, as the nature of the actual demands which are made upon the graduates whom we turn out. The problem of the school is to meet the demand for men who can do something, men who can think in a logical and common sense way, men who are ready to accomplish results from the outset. The demand is not for highly trained mathematicians (however much we may think it is) but for men who can use a little mathematics and use it effectually as a tool. The danger is that the mathematical requirement may be too high. The mathematical concepts are difficult and it may be that we are expecting too much, too varied, and extensive attainments. The great thing is to gain ability to think correctly and logically about things, and in this respect the teacher of mathematics and the teacher of engineering have the same task before them. There is no better place than the mathematical class-room to develop logical thinking, but it should be done in connection with well selected concrete problems, rather than in the realm of abstract and theoretical considerations. The man who can take hold of a problem, analyze the data, put together the facts and reason logically to a sound conclusion, is already on the road to a successful career, and the teacher who is helping his students to attain such power is fulfilling his mission to the student who is to prepare for engineering.

Mr. Waddell refers to the teaching of mathematics to engineering students twenty years ago as sufficiently strenuous but far from satisfactory and finds little indication of radical improvement at the present time. He holds that the engineering student in his pure mathematical classes is not taught what the equations employed really mean, but that much of his work is a juggling with quantities to produce certain results, while it is left to the teacher of rational mechanics to bring out the reality of mathematics. He would advocate greater emphasis upon the interpretation of symbols used, the constant resort to graphical methods, expecially in analytics, the concrete application of all principles developed, especially in the calculus, a greater attention to the subject of descriptive geometry early in the engineering curriculum, and a thorough mastery of mechanics, whose foundation is mathematics and which underlies the superstructure of engineering. Mr. Waddell is strongly opposed to the lecture system of teaching mathematics to engineering students. Aside from introductory and concluding lectures, together with informal talks from time to time on the general trend of the subject, he would insist on the midnight oil and the damp towel as the necessary accessories for obtaining a real and comprehensive grasp of the subject for the average student. As to the instructional staff in engineering schools he would insist that the teachers of mathematics should also be engineers and should have a practical acquaintance with both rational and practical mechanics. And finally as to the extent of mathematical study for an engineer, while it is true that the actual use of analytic geometry, calculus, least squares, etc., rarely occurs in the practice of most engineers, yet the engineer's grasp of technical work depends upon his knowledge of these subjects and hence his mathematical foundation must be strong and substantial.

The general discussion was supported by Mr. C. F. Scott, Pittsburg, Pa., Dean C. M. Woodward, Washington University, Professor B. F. Groat, School of Mines, University of Minnesota, Professor S. M. Barton, University of the South, President C. S. Howe, Case School of Applied Science, Professor C. A. Waldo, Purdue University, Professor C. B Williams, Kalamazoo College, Mr. J. B. Webb, consulting engineer, Hoboken, N. J., Dean H. T. Eddy, College of Engineering, University of Minnesota, Professor D. F. Campbell, Armour Institute of Technology, Professor A. E. Haynes, College of Engineering, University of Minnesota, Professor E. W. Davis, University of Nebraska, Professor A. S. Hathaway, Rose Polytechnic Institute, and Professor E. V. Huntington, Harvard University.

Abstracts of these shorter addresses are not immediately available, but doubtless some of the more important utterances may be printed in connection with the formal papers in *Science*.

H. E. SLAUGHT,
Secretary of the Chicago Section.

THE FIFTY-EIGHTH MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

THE fifty-eighth meeting of the American Association for the Advancement of Science was held at the University of Chicago during the convocation week December 30, 1907, to January 4, 1908. The president of the meeting was Professor E. L Nichols, Cornell University. The address of the retiring president, Dr. W. H. Welch, entitled "The interdependence of medicine and other sciences of nature," was given in the Leon Mandel Assembly Hall on the evening of the opening day.

The meetings of Section A were of unusual interest in view of enthusiastic joint sessions with Section D and the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY to consider the present status and means of improvement of the teaching of engineering students of elementary mathematics, and also to listen to the address of the retiring vice-president, Professor Edward Kasner, of Columbia University, entitled "Geometry and mechanics." The officers of the section were: vice-president, E. O. Lovett; secretary, G. A. Miller; councilor, G. B. Halsted; member of the general committee, F. R. Moulton; sectional committee, Edward Kasner, E. B. Frost, E. O. Lovett, Harris Hancock, F. R. Moulton, E. W. Brown, and G. A. Miller. The following program was presented at the separate meetings of Section A:

(1) Professor E. B. FROST: "Observations with the Bruce spectrograph."

(2) Professor E. B. FROST: "Comments on the Zeiss stereocomparator and the spectrocomparator belonging to the Yerkes observatory."

(3) Professor E. E. BARNARD: "Photographic phenomena of comet *d*, 1907 (Daniel)."

(4) Professor E. E. BARNARD: "On a great bed of nebulosity in Sagittarius, photographed with the Bruce telescope of the Yerkes observatory."

(5) Professor E. O. LOVETT: "Note on the problem of three bodies."

(6) Professor JOEL STEBBINS: "The light curve of Delta Cephei."

(7) Professor JOEL STEBBINS and Mr. F. C. BROWN: "An application of the selenium cell to astronomical photometry."

(8) Professor E. E. BARNARD: "Observations and explanation of the phenomena seen at the disappearances of the rings of Saturn."

(9) Professor E. B. FROST and Mr. PHILIP FOX: "On the twenty-foot horizontal solar spectrograph of the Yerkes observatory."

(10) Professor KURT LAVES: "A graphic method for the determination of the orbit of a spectroscopic binary."

(11) Professor KURT LAVES: "New tables for the time of sight correction of the earth's orbital motion."

(12) Professor W. D. CAIRNS: "A generalized theory of integral equations."

(13) Professor G. B. HALSTED: "On the theory of order, static and nascent."

(14) Dr. F. W. REED: "Singular points in the approximate development of the perturbative function."

(15) Professor JOHN EIESLAND: "On a certain class of algebraic translation surfaces."

(16) Dr. ARTHUR RANUM: "Matrices not belonging to groups."

(17) Professor C. H. BECKETT: "A note on interest on reserve from items computed for the uniform report blank."

(18) Dr. A. E. YOUNG: "On asymptotic isothermic surfaces."

(19) Professor G. A. MILLER: "Some questionable terms and definitions used in elementary mathematics."

(20) Messrs. J. A. PARKHURST and F. C. JORDAN: "The photographic determination of star colors and their relation to spectral type."

(21) Mr. PHILIP FOX: "On the detection of the eruptive prominences on the solar disk."

(22) Mr. PHILIP FOX: "An investigation of the 40-inch objective at the Yerkes observatory."

(23) Mr. R. J. WALLACE: "The function of a color filter and of certain plates in astronomical photography."

(24) Professor G. E. HALE: "The vertical coelostat or 'tower' telescope of the Mt. Wilson solar observatory."

(25) Professor G. E. HALE and Mr. W. S. ADAMS: "Preliminary results of a comparative study of the spectra of the limb and the center of the sun."

(26) Professor FRANK SCHLESINGER: "A simple method for reducing spectrograms."

(27) Professor F. R. MOULTON: "On the probability of the near approach of two stars and on relative problems in the sidereal universe."

In the absence of their authors the papers by Professors Cairns, Halsted and Lovett and Dr. Ranum were read by title; that by Professor Schlesinger was presented by Mr. Fox. All the other papers in the above list were read by their authors. The following abstracts of those which were of mathematical interest bear numbers corresponding to those of the titles in this list. The abstracts of the others appeared in the report of this meeting published in *Science*, January 31, 1908.

5. Professor Lovett constructs a problem of three bodies possessing exact transcendental solutions defined by finite equations; these solutions reduce to the lagrangian solutions of the classic problem of three bodies when the ideal problem assumes the newtonian form. The paper is a part of an extended memoir which is not yet ready for publication.

12. By a generalization of the Hilbert theory of a quadratic form in an infinite number of variables, Professor Cairns gives a complete treatment for the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\phi(t)dt + x'g(s)$$

with the auxiliary equation

$$\int_a^b g(t)\phi(t)dt = c,$$

$f(s)$, $g(s)$, and c being given, $K(s, t)$ a generalized Green's function, and $\phi(s)$ the required function.

Application is made to the differential equation

$$p(x)\frac{d^2u}{dx^2} + \frac{dp}{dx}\frac{du}{dx} + q(x)u + \lambda' g(x) + \phi(x) = 0$$

with the auxiliary equation

$$\int_a^b u(x)g(x)dx = 0,$$

to a generalized Fourier development, and to the isoperimetric problem

$$\int_a^b F(y', y, x)dx = \text{minimum},$$

given

$$\int_a^b G_i(y', y, x)dx = l_i \qquad (i = 1, 2, \cdots, m).$$

13. The paper by Professor Halsted investigates the essence of inherent order, its foundation and genesis, and considers the uses of inherently ordered series in the attribution of factitious order to primarily unordered assemblages. It gives the solution of problems in arrangement and investigates the betweenness relations, linear and of more than one dimension. It gives applications to the theory of teaching, to arithmetic, and to geometry.

14. The method of approximation developed by Poincaré for obtaining the terms of higher order of the perturbative function depends uniquely upon certain singularities of this function. In the general case the singular points are given by algebraic equations of a high degree. The discussion relative to the actual admissibility of these points is often very delicate, but a criterion once established leads to the solution of the general case by extension of the results found by Poincaré, Hamy, Feraud, and Coculesco in certain restricted cases. With suitable numerical assumptions the case of small eccentricities and a small inclination of the orbits was carried out in detail by Dr. Reed, and the results to be found by varying the elements are indicated. The method is especially applicable to the problem of finding the values of the coefficients with small divisions in the perturbations of the small planets by Jupiter.

15. Professor Eiesland's paper contains a treatment of all the types of translation surfaces that are determined by a unicursal quartic in the plane at infinity. The most important result obtained is this: To a unicursal quartic with three real double points correspond ∞^3 types of translation surfaces of the form

$$(1) \quad A + Be^X + Ce^Y + De^Z + Ee^{X+Z} + Fe^{X+Y} + Ge^{Z+Y} + He^{X+Y+Z} = 0$$

where the coefficients satisfy the following identical relation:

$$EGAF = HDCB.$$

By a logarithmic transformation

$$x = \log X, \quad y = \log Y, \quad z = \log Z$$

this surface is converted into the cubic surface

$$(2) \quad A + Bx + Cy + Dz + Exz + Fxy + Gzy + Hxyz = 0$$

which is the exact analog of the surface

$$Ax + By + Cz + Dxy + Exz + Fzy = 0$$

obtained by Lie for the case of a degenerate quartic consisting of two conics.* It is the intention of the writer to study the surfaces (2) more in detail. The paper will be offered to the *American Journal of Mathematics* for publication.

16. All ordinary n-ary matrices (linear substitutions with non-zero determinants) form a group, and some singular matrices (with zero determinants) also belong to groups. If S is a singular matrix not belonging to any group, then if the number of zero roots of its characteristic equation is s, there always exists a positive integer $m \leqq s$ such that S^m belongs to some group. If m is the lowest such integer, then among the invariant factors of the characteristic determinant of S that correspond to zero roots there is at least one of order m and none of higher order. Dr. Ranum also showed that if S is of rank r, it is reducible to an ordinary $(n-s)$-ary partial matrix and $n-r$ singular, irreducible partial matrices, each of which is nilpotent, *i. e.*, has a power (exponent $\leqq m$) equal to zero.

17. By an agreement among the insurance commissioners and supervisory departments of a majority of the States of the Union, a detailed and comprehensive report of the year's business is required of life insurance companies on a form known as the union form report blank. In the "profit or loss" account of this sheet there is an item "interest required to maintain reserve." The net income from investments is exhibited and from these items the "profit or loss" from interest earnings is shown. There is a discrepancy between the mathematical theory involved in the fundamental calculations, and the policy contract itself which leaves some latitude and consequent confusion. The purpose of Professor Beckett's paper is to set forth this problem and find a solution that can be checked from the other

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*See Lie-Scheffers, Berührungstransformationen, vol. 1, pp. 387 and 410.

items required to be computed and exhibited in the report, in the general interest of a fair understanding and uniformity: All life insurance calculations are based upon the theory that premiums are paid at the beginning of the insurance year, and death claims paid at the end of the year. All modern life insurance contracts provide that payment shall be made upon satisfactory proofs of death, or in a few cases within 60 days thereafter.

The death strain or "mortality cost" in any case is the amount at risk multiplied by the probability of dying, *i. e.*, $C_{x+m}(S - V_{x+m})q_{x+m}$. The mean reserve which appears in the report blank is $\frac{1}{2}(P_x + V_{x+n} V_{x+n+1}) = m_{x+n+\frac{1}{2}}$. The net cash value of the policy at the end of any year, or terminal reserve, is the fund which together with the net premiums to be received in the future is the exact mathematical equivalent of the obligation incurred by the company to pay the contract.

In any individual case, interest $3\frac{1}{2}$ %,

$$1.035\,V + P_x(1.035) = V_{x+n+1} + C_{x+n},$$

$$M_{x+n+\frac{1}{2}} = \frac{V_{x+n} + P_x + V_{x+n+1}}{2}, \quad V_{x+n+1} = 2M_{x+n+\frac{1}{2}} - V_{x+n} - P_x,$$

$$V_{x+n} = \frac{2M_{x+n+\frac{1}{2}} - V_{x+n} - P_x + C_{x+n+1} - P_x(1.035)}{1.035},$$

$$V_{x+n} = \frac{M_{x+n+\frac{1}{2}} + \frac{1}{2}C_{x+n+1} - P_x(1.0175)}{1.0175}.$$

Let

$$V_{x+n} + P_x = R, \quad R = \frac{M_{x+n+\frac{1}{2}} + \frac{1}{2}C_{x+n+1}}{1.0175}.$$

Interest for the first half year

$$R(.0175) = \frac{.0175}{1.0175}\left[M_{x+n+\frac{1}{2}} + \tfrac{1}{2}C_{x+n+1}\right].$$

Let $V_{x+n-1} + P_x = K$.

Interest for the second half year

$$K(.0175) = \frac{.0175}{1.0175}\left[M_{x+n+\frac{3}{2}} + \tfrac{1}{2}C_{x+n+2}\right].$$

If from these sums there be deducted $3\frac{1}{2}$ per cent. of the mean net deferred premiums, the result gives the interest required to maintain reserve in terms of other items which are reported in the blank. This is a close check and true in theory, but not quite true in actual practice and contract. In case a death claim falls in the early part of the insurance year on a contract that is young, the loss is conspicuous. The only way to make a perfect agreement between theory and contract would be to recompute all net premiums and all other tables involved in the calculations. Upon the new assumption that claims are paid immediately, this is possible but not practical, as it would cause great confusion, and since it could not be retroactive, it would take a long period of years to have all forms in conformity to this practice. In the meantime great confusion would prevail. Should the company earn only this guaranteed rate of interest upon net premiums and reserves, there would be a deficiency of approximately .0035 per cent. on a mixed business of paid-for insurance, endowment, and limited payment forms.

Claims cannot be paid actually at the moment of death. Statistics show that the actual necessary lapse of time averages one month and that on an average the loss of interest upon experienced mortality funds paid out is five months. The best practical adjustment would then be to discount this fund upon this five-month basis and require the company to make good the deficiency out of excess earnings over the rate assumed in the contract.

18. Surfaces characterized by having isothermal asymptotic lines and isothermal lines of curvature Dr. Young has called "asymptotic-isothermic," and in a paper read before the Chicago section of the AMERICAN MATHEMATICAL SOCIETY, March 30, 1907, he outlined a general method for the determination of these surfaces and discussed a particular solution of the problem which led to certain surfaces whose first fundamental forms, when referred to lines of curvature, come under the more general form

$$\overline{ds}^2 = \frac{(u+v)^k}{(u-v)^l}\left(\frac{du^2}{U} + \frac{dv^3}{V}\right),$$

where k and l are constants, and U and V are functions of u and v respectively. He continues the discussion in this paper and determines all surfaces which have the above expression for their first fundamental form when referred to lines of curvature, and also all isothermic surfaces which have the same spherical representation of their lines of curvature as these. All of the new surfaces are asymptotic-isothermic, with the exception of one class composed of an infinity of surfaces.

19. The term division has two distinct meanings in elementary mathematics.* According to one of these, it implies the operation of finding an integral quotient and an integral remainder, while according to the other it implies the finding of a number which multiplied into the divisor produces the dividend. While only the latter is the inverse of multiplication, yet it is customary to speak of division as an inverse operation without specifying which of the two commonly accepted definitions of the term is meant. A very common definition of multiplication is the performing upon the multiplicand the same operation as that which is performed upon unity to get the multiplier. The vagueness of this definition follows directly from the fact that 4 may be obtained by doubling unity and squaring the result, yet multiplying by 4 does not generally mean doubling the multiplicand and squaring the result. Such vague definitions are contrary to the very essence of mathematics and hence should be avoided.

Especial stress was laid by Professor Miller upon the fact that dividing by 0 should be banished from elementary mathematics. The so-called indeterminate forms are really meaningless forms and it is questionable whether one should speak of evaluating such a form. As $uv = 0$ has not always for its locus the combined loci of $u = 0$ and $v = 0$, the rule relating to this case should be stated with the necessary restrictions. The fact that the last letters of the alphabet are used both for variables and for unknowns in elementary algebra has led some authors to speak of these two generally distinct concepts as if they were identical. This is the more unfortunate since the concept of a variable is continually playing a more fundamental rôle in elementary algebra. The paper has been offered to *School Science and Mathematics* for publication.

27. The problem of Professor Moulton's paper is to find the possible rôle that the near approach of the stars to one another

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* Cf. Encyclopédie des Sciences mathématiques, vol. 1 (1904), p. 43.

may have played in sidereal evolution. The answer to this problem depends upon the extent of the sidereal universe, the number of stars in it, and the character of their motion. The discussion leads to the conclusion that the relatively near approaches of the stars have probably been an important factor in stellar evolution. This paper will be published by the Carnegie Institution of Washington.

The next regular meeting of the association will be held at Johns Hopkins University under the presidency of Professor T. C. Chamberlin, of Chicago University. A summer meeting will be held at Dartmouth College, beginning the last Monday in June. Professor C. J. Keyser, Columbia University, was elected vice-president and chairman of Section A, and Professor G. A. Miller, University of Illinois, was reelected secretary. The former election is for one year and the latter for a period of five years in accord with the general rules of the association. A resolution of the council which is of especial interest to the various scientific societies is that sectional committees may dispense with a sectional program whenever an affiliated society covering the same scope meets with the section. This resolution was reaffirmed to avoid duplications of programs and to secure more hearty cooperation between the scientific societies and the association.

G. A. MILLER,
Secretary.

UNIVERSITY OF ILLINOIS.

SHORTER NOTICES.

Memoir and Scientific Correspondence of the late Sir George Gabriel Stokes, Bart., Sc.D., etc. Selected and arranged by JOSEPH LARMOR, D.Sc., LL.D., Sec. R.S. Two vols. 8vo. Cambridge, at the University Press.

THE history of mathematics and physics during the latter half of the nineteenth century is inseparably bound up with the career of Sir George Gabriel Stokes. As an investigator of the highest rank himself and as one who left a permanent mark on his own science, the story of his life and career is a necessary part of the development of the subject. But it is much more than this. His position as secretary of the Royal society for over thirty years and as president for five years gave him unusual opportunities to become acquainted with those who were active in scientific work generally, and to keep in touch with all the scientific progress of his time. Of these opportunities he made the best possible use, not for himself or for his own interests, but in assisting those who wished to consult him or to obtain his advice. He was absolutely unselfish in his willingness to give his time and thought to all who wished to obtain assistance from him, and as even his first opinions on any debatable point were rarely wide of the mark, and as he would nearly always throw out some valuable suggestion or idea, the influence which he exerted cannot be easily overestimated.

Professor Larmor undertook the difficult task of gathering together and publishing the materials which should show properly these features of Stokes's character and at the same time record the inner scientific history of the time; there can be no doubt that he has achieved a notable success. The former are set forth partly in a memoir by Mrs. Laurence Humphry, the daughter of Sir George Stokes, and partly in appreciations by Professor Liveing, Sir Michael Foster, Sir William Huggins, and the Right Rev. Bishop G. F. Browne. Though the bias which necessarily accompanies the point of view of one writer is thus avoided, one cannot help wishing that the editor, as one well fitted to judge and as belonging to a later generation, had himself contributed a notice instead of confining his share of the work to selection from the materials at his disposal and to the filling in of such details as were necessary to make them intelligible or clear. The latter object, a record of the inner scientific history of the time, has been mainly achieved by the publication of numerous letters to and from various correspondents.

In the first volume the arrangement of these letters is mainly chronological, but the subjects dealt with in them are kept together as far as possible. With the careful selection of headlines the editor has made it easy for a reader to obtain the main outlines of Stokes's scientific career by merely turning over the pages. A full index enables him to find those parts which deal with any particular subject. In the second volume we are given almost entirely letters to and from correspondents arranged under the names of the latter. A striking omission — the absence of letters between Lord Kelvin and Stokes — is explained in the preface. We are glad to learn that they suffice to form a collection by themselves and that it has been decided to publish them separately with a memorial of the lifelong friendship and collaboration of the writers.

The memoir by Mrs. Laurence Humphry is full of interesting details. In a letter to Lady Stokes before their marriage we learn how the notable paper, "On the discontinuities of the arbitrary constants which appear in divergent developments" was finally brought to a successful conclusion. Stokes writes, "I have been . . . sitting up till 3 o'clock in the morning fighting hard against a mathematical difficulty. Some years ago I attacked an integral of Airy's and after a severe trial reduced it to a calculable form. But there was one difficulty about it, which, though I tried till I almost made myself ill, I could not get over, and at last I had to give it up and profess myself unable to master it. I took it up again a few days ago, and after a two or three days fight, the last of which I sat up till 3, I at last mastered it."

In speaking of these same letters Mrs. Humphry writes, "They are remarkable also from the curious place which he assigned to his original investigations; it almost seems as if he considered them the height of dissipation and everything else a duty." One might quote almost at random from the letters to his numerous correspondents and give something worthy of notice. A considerable number of them are scientific papers of the highest value, although the results have of course been incorporated in the published memoirs of the time. But we may gather much from the tentative suggestions which may be made in a private letter but which one might not feel justified in publishing in a scientific paper. Whether we read through these volumes continuously or dip into them here and there, their perusal cannot fail to be of interest and value.

E. W. BROWN.

Computation and Mensuration. By P. A. LAMBERT. The Macmillan Company, New York, 1907. ix + 92 pp.

THIS little volume is intended to help bridge the gap which frequently exists between the mathematics of the secondary schools offering advanced algebra and trigonometry for entrance to college and college mathematics proper. In colleges where algebra and trigonometry are taught as a part of the regular curriculum in mathematics the book might very readily be used as a review text preparatory to the work in the calculus. Opportunity to review and apply mathematical principles learned is given by a set of 165 concrete problems of a grade which insures the arrival at definite results. The selection of the problems is based on the correct pedagogic principles that doing things promotes and sustains the students' interest, and that all work in mathematics should be consecutive.

That the text is written from the view-point of the instructor whose students are preparing for technical courses may readily be seen from the selection of the topics discussed. The following form a list representative of the contents of the ten chapters: Measurements, both direct and indirect, of lengths and areas; with special reference to the degree of accuracy possible in the computations based on the given data. Areas of rectilinear plane figures by the method of coordinates, with applications to problems in surveying. The prismoidal formula, with special reference to the computation of earthwork. Vectors, with applications to resultant forces and latitudes and departures in surveying. A chapter on the nature and applications of logarithms, which might have been strengthened by a more complete discussion of the slide rule and by the introduction of problems based on the system of natural logarithms. Approximations to the lengths of curved lines as problems in limits. Graphic algebra, including work in the solution of simultaneous equations and inequalities. Approximate areas, leading to the trapezoidal, Simpson's, and Weddle's formulas. Approximate volumes, with applications to the contents of vessels. A discussion of the results obtained by the use of the prismoidal formula when the volumes are those of revolution.

Throughout it is apparent that the author believes in the pedagogic value of accuracy, the proper arrangement of computations, and the intelligent interpretation of processes and results. It seems to the reviewer that, since the nature of the subjects considered holds the student continually responsible for concrete results, the answers to the problems should have been included in the text.

ERNEST W. PONZER.

Anfangsgründe der darstellenden Geometrie für Gymnasien. Von FRITZ SCHÜTTE, Oberlehrer am Gymnasium zu Düren. Leipzig, Teubner, 1905. 42 pp.

SINCE instruction in descriptive and constructive geometry was introduced into the German gymnasia in 1901, a number of elementary texts have appeared, designed to meet the needs of the new class of pupils in this subject. The present book begins with a detailed concrete explanation of the elements of descriptive geometry. All definitions are put in bold-faced type. Solid bodies are depicted on the ground, vertical, and profile planes, and various objects are drawn to show that all three elevations are generally necessary. The straight line, plane, prism, pyramid, and regular bodies are treated in succession, a number of unsolved exercises being appended to each case. Then follow the circle, cylinder, cone, and sphere. After these ideas are mastered, the pupil is prepared to take up parallel perspective, wherein the same figures are treated again, but rather more concisely. Finally, a few pages are devoted to central perspective; methods are given for constructing a perspective picture of an object when its ground plan and profile are given. The author explains in the preface that the arrangement of subject matter is new, that heretofore too much emphasis has been laid on perspective drawing. One is tempted to feel that he has possibly gone to the other extreme, but at any rate the intelligent reader has not been bored by a multitude of details, and will finish the book thirsty for more. The typographical work is excellent; the figures are crude, but easily understood. The usefulness of this little book should not be confined to the German gymnasia.

VIRGIL SNYDER.

Petrus Peregrinus de Maricourt and his Epistola de Magnete. By SILVANUS P. THOMPSON, D.Sc., F.R.S. From the *Proceedings of the British Academy*, Volume II. London, published for the British Academy by Henry Frowde, Oxford University Press. 32 pp.

FOR thirty years Professor S. Thompson has made a study of the early history of magnetism, whenever opportunity presented itself for the examination of original sources. A result of this study is this article on Petrus Peregrinus and his famous Epistle on magnetism that was "done in camp at the siege of Lucera, Anno Domini 1269, the eighth day of August." No student of the history of science in the time of Peter Peregrinus and Roger Bacon can afford to overlook this article of Professor Thompson. He gives a list of the twenty-eight different ancient manuscript copies of the Epistle, tells where each is found and compares them. Two of these are in his own possession, the rest are in public libraries. "No two have yet been found to agree precisely in their text." The manuscripts in Erfurt, Vienna, Dublin, and Geneva have not yet been critically examined, hence it is still doubtful "what readings ought to be finally adopted for all the doubtful passages."

Of printed versions there are nine, or eleven if one includes a plagiarized work of 1562 and its English translation of 1579. The earliest printed version is the Augsburg tract of 1558, of which only eighteen copies are now known to exist, one of these being in the library of the American institute of electrical engineers in New York City.

FLORIAN CAJORI.

Traite des Assurances de la Vie. By U. BROGGI. Translated from Italian into French by S. LATTÈS. Paris, Hermann, 1907. xi + 306 pp.

THE appearance of Professor Broggi's book in a French translation makes it accessible in three languages, an edition in German having already appeared. This fact in itself is significant and bespeaks an examination of the volume.

If anyone has been accustomed to regard a text on the theory of life insurance as an unattractive compendium of formulas not all of which are carefully derived, to such a one the volume under review will by contrast be welcome. For the author has elected to present the subject as an integral part of mathematical theory, and has exhibited a proper concern regarding the development of the doctrine of probability which is the basis of the subject. He has also applied himself to the clear formulation of the various special problems which are treated and to the use of the mathematical processes in valid ways only. Even at this day successful undertakings of this sort are not so common as to be unworthy of praise.

An introductory statement of the problem of insurance is followed by the four chief divisions of the book; and of these the first contains a good discussion of the more interesting elementary topics of the theory of probability, including the theorems of Bernoulli and Poisson, an introduction to the method of least squares and a few formulas on interest. The second and third parts, constituting a considerable portion of the book, are devoted to the derivation of formulas for the principal forms of annuities and insurance upon groups of lives, the laws of mortality of De Moivre, Gompertz, and Makeham furnishing the basis of the discussion, and for the premiums and reserves of insurance policies. Naturally these parts are devoted mainly to matters which must be treated in all complete texts on insurance. A shorter fourth part treats the theory of risk.

Good judgment has been shown by the author both in the selection of topics and in the apportionment of space to them. Not only will the book be of particular interest to those who desire to deal with the theory of life insurance as one phase of the mathematics of statistics, but it will prove instructing to most and valuable to all of those who deal primarily with actuarial science.

G. H. LING.

CORRECTION.

PROFESSOR H. W. Kuhn has informed me that the theorem relating to the totality of the substitutions which are commutative with every substitution of a transitive group, published in my article in the present volume of the BULLETIN, page 19, is not new. It is found in Professor Kuhn's doctor dissertation, *American Journal of Mathematics*, volume 26 (1904), page 67.

G. A. MILLER.

NOTES.

THE Annual Register of the AMERICAN MATHEMATICAL SOCIETY for 1908, containing the list of officers and members of the Society, constitution and by-laws, annual reports, and catalogue of journals in the Society's library together with other accessions for 1905–1907, has recently been issued. A complete catalogue of the library up to 1905 is contained in the Register for that year, copies of which may still be obtained from the Secretary.

THE January number (volume 9, number 2) of the *Annals of Mathematics* contains the following papers: "On the classification of plane algebraic curves possessing fourfold symmetry about a point," by R. D. CARMICHAEL; "A second inverse problem in the calculus of variations," by C. E. STROMQUIST; "The continuous plane motion of a liquid bounded by two right lines," by H. C. WOLFF; "A problem in chance," by J.

K. WHITTEMORE; "The expression of constant and of alternating continued fractions in hyperbolic functions," by A. E. KENNELLY.

THE third regular meeting of the Rochester section of the association of teachers of mathematics in the middle states and Maryland was held at the University of Rochester, Saturday, February 8. The following papers were read: By W. BETZ, "The use of intuition in geometry"; by H. D. MINCHIN, "Physical demonstrations for the mathematical class-room"; by J. H. TANNER, "Genesis and extension of the number concept"; by Miss K. L. CARING, "Discussion of the syllabus of elementary algebra." The next meeting will be held at the University of Rochester on Saturday, April 11.

AT the meeting of the London mathematical society held on January 9, 1908, the following paper was read: By C. S. JACKSON, "A formula of interpolation."

THE following programme for the Fourth international congress of mathematicians, to be held at Rome in April, is announced:

Sunday, April 5, 9:30 P. M. General reception in the Aula of the University of Rome.

Monday, 10 A. M. Opening of the congress at the Capitol. 3 P. M. Election of the general bureau; award of the Guccia medal. First and second conferences.

Tuesday, Wednesday, Thursday, Friday, Saturday. 9 A. M. Sectional meetings. 3 and 3:30 P. M. Two conferences.

Organization of the sections:

I. Arithmetic, algebra, analysis.: opened by ARZELÀ, CAPELLI, PASCAL, PINCHERLE.

II. Geometry: BIANCHI, SEGRE.

III. Mechanics and applied mathematics: LEVI-CIVITA, LUIGI, PIZZETTI, TOJA.

IV. Philosophic, historic, and pedagogic questions: ENRIQUES, LORIA, VAILATI.

All meetings after the opening one will be held at the Royal academy dei Lincei, Via della Lungara, 10. Reduced rates (40 to 60 per cent.) will be offered on all Italian railroads and in many hotels in Rome. All reports, conferences, and formal papers read at the Congress will be collected in the volume of the proceedings, to be published by a committee under the management of the director of the *Rendiconti del Circolo matematico di Palermo.* Authors are requested to give their manuscripts to the general secretary before the close of the meetings.

The membership fee of 25 lire should be sent to Professor Vincenzo Reina, treasurer, Piazza S. Pietro in Vincoli 5, Rome, not later than March 25.

Subjects of conferences:

DARBOUX: "Quelques idées dans la géométrie infinitésimale."

FORSYTH: "On the present condition of partial differential equations of the second order, as regards formal integration."

HILBERT: "Die Methode der unendlich vielen unabhängigen Variabeln."

KLEIN: "Über die mathematische Encyklopädie."

LORENTZ: "Le partage de l'énergie entre la matière pondérable et l'éther."

MITTAG-LEFFLER: "Sur la représentation arithmétique des fonctions analytiques générales d'une variable complexe."

NEWCOMB: "La théorie du mouvement de la lune; son progrès et son état actuel."

PICARD: "L'analyse dans ses rapports avec la physique mathématique."

POINCARÉ: (Subject not yet announced).

VERONESE: "La geometria non archimedea."

In the inaugural session, Professor VOLTERRA will discuss "Le matematiche in Italia nella seconda metà del secolo XIX."

AT the annual public meeting of the Paris academy of sciences, held on December 2, 1907, the following prizes in mathematical sciences for 1909 and 1910 were announced; competing memoirs should be in the hands of the secretary before December 31 of the year preceding that in which the respective prizes are awarded: Francoeur prizes (fr. 1000) will be awarded annually for general progress in pure and applied mathematics; Bordin prize (fr. 3000) for 1909, see BULLETIN, volume 13, page 308; Grand prize (fr. 3000) for 1910, for the solution of the following problem: "It is known how to find all the systems of two meromorphic functions in the plane of the complex variable, connected by an algebraic relation. An analogous question arises for a system of three uniform functions of two complex variables, each having the character of a rational function in the finite part of the field, and all connected by an algebraic function. In default of a complete solution of the problem, the academy at least desires the treatment of particular examples, leading to classes of new transcendentals."—Fourneyron prize (fr. 1000) for 1910: "Theoretic and experimental study of the blow of a battering ram in an elastic tube."—Poncelet prize (fr. 2000) for 1909 for general progress in applied mathematics; for 1910, for an important memoir in pure mathematics.—Vaillant prize (fr. 4000) for 1909: "Perfect, in some important point, the application of the dynamical principles of fluids to the theory of the propeller."—Boileau prize (fr. 1300) for 1909, for an important contribution to the theory of fluids. The remaining prizes will be awarded under the same conditions as those previously announced.

Dr. H. Lebesgue has been promoted to a professorship of mathematical analysis at the University of Poitiers.

The late Mr. Richard Brown, of Youngstown, Ohio, left the sum of $30,000 to Mount Union College to endow the chair of mathematics. The present professor, Mr. B. F. Yanney, has been appointed to the chair.

Professor C. O. Gunther, of the Stevens Institute of Technology, has been promoted to be acting professor of mathematics.

Professor W. W. Beman, of the University of Michigan, has been granted a leave of absence during the next academic year.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Baker (H. F.). Introduction to the theory of multiply periodic functions. Cambridge, 1907. 8vo. 16 + 336 pp. Cloth. 12s. 6d.

Bôcher (M.). Introduction to higher algebra; prepared for publication with the cooperation of E. P. R. Duval. New York, Macmillan, 1908. 8vo. 11 + 321 pp. Cloth. $1.90

Bromwich (T. J. I'A.). Quadratic forms and their classification by means of invariant factors. (Cambridge tracts in mathematics and mathematical physics.) New York, Putnam, 1907. 8vo. 8 + 100 pp. $1.10

Brozat (W.). Ueber Scharen von ∞^4 Flächen im R_3, die durch Berührungstransformationen in Scharen von ∞^4 Kurven überführbar sind. Greifswald, 1907. 8vo. 53 pp. M. 1.80

BRUNO (K.). Die Grundlehren der Integral- und Differentialrechnung. Wien, Hölder, 1908. 8vo. 51 pp. M. 1.25

BUSTELLI (A. M.). Elementi di filosofia della matematica nei riguardi didascalici, con prefazione di V. Cerruti. Fasc. 1–4. Roma–Milano, Società ed. Dante Alighieri, 1905–07. 8vo. 4 fasc. Pp. 46; 54; 13 + 40; 82. L. 2.80

CERRUTI (V.). See BUSTELLI (A. M.).

CORNACCHIA (G.). Su di un metodo per la risoluzione in numeri interi dell' equazione $\sum_{k=0}^{n} C_k x^{n-k} \cdot y^k = P$. Faenza, Montanari, 1907. 4to. 15 pp.

DONADT (A.). See LÜBSEN (H. B.).

DUVAL (E. P. R.). See BÔCHER (M.).

GALILEI (G.). Unterredungen und mathematische Demonstrationen über zwei neue Wissenszweige, die Mechanik und die Fallgesetze betreffend. Aus dem Italienischen übersetzt und herausgegeben von A. von Oettingen. (Ostwalds Klassiker der exakten Wissenschaften, No. 11.) 2ter unveränderter Abdruck. Leipzig, Engelmann, 1907. 8vo. 142 pp. M. 3.00

JUNKER (F.). Repetitorium und Aufgabensammlung zur Differentialrechnung. 2te Auflage. Neudruck. Leipzig, 1907. 12mo. 129 pp. Cloth. M. 0.80

KELLER (S. S.). Analytical geometry and calculus. New York, Van Nostrand, 1907. 12mo. 359 pp. Cloth. $2.00

LARMOR (J.). See POINCARÉ (H.).

LÜBSEN (H. B.). Ausführliches Lehrbuch der analytischen oder höheren Geometrie, zum Selbstunterricht bearbeitet. 15te Auflage, neu bearbeitet von A. Donadt. Leipzig, 1907. 8vo. M. 4.00

OETTINGEN (A. VON). See GALILEI (G.).

PAGLIERO (G.). Applicazioni del calcolo infinitesimale. Torino, Paravia, 1907. 8vo. 244 pp. L. 7.00

PERCIVAL (A. S.). Practical integration. For the use of engineers, etc. London, Macmillan, 1907. 8vo. 92 pp. 2s. 6d.

POINCARÉ (H.). Science and hypothesis; with a preface by J. Larmor. New York, Scribner, 1907. 12mo. 27 + 244 pp. Cloth. $1.25

SEAVER (E. P.). Mathematical handbook; containing the chief formulas of algebra, trigonometry, circular and hyperbolic functions, differential and integral calculus, and analytical geometry; together with mathematical tables. New York, McGraw, 1907. 8vo. 10 + 278 pp. Cloth. $2.50

SITTIGNANI (M.). Sulle funzioni intere di genere finito. Bologna, Zanichelli, 1907. 8vo. 8 pp.

VALENTINER (S.). Vektoranalysis. Leipzig, 1907. 12mo. 163 pp. Cloth. M. 0.80

II. ELEMENTARY MATHEMATICS.

AMANZIO (D.). Trattato di algebra elementare. 3a edizione. Napoli, Pellerano, 1907. 8vo. 543 pp. L. 4.50

BARBISCH (H.). See JAHNE (J.).

Bremiker (C.). Tavole logaritmico-trigonometriche con cinque decimali. Edizione italiana eseguita per cura di L. Cremona. Quinta tiratura stereotipa, con una prefazione e tavole dell' ing. A. Salmoiraghi. Milano, Hoepli, 1908. 8vo. 47 + 161 pp. L. 2.50

Briggs (W.) and Bryan (G. H.). Tutorial trigonometry. 2d edition. London, Clive, 1907. 8vo. 332 pp. Cloth. 3s. 6d.

Bryan (G. H.). See Briggs (W.).

Buchanan (A. H.). Plane and spherical trigonometry. New York, Wiley, 1907. 8vo. 6 + 96 pp. Cloth. $1.00

Cremona (L.). See Bremiker (C.).

Éléments de géométrie. Lignes droites; perpendiculaires et obliques; parallèles; circonférence; mesures; lignes proportionnelles; figures; polyèdres, etc. Nouvelle édition. Paris, Boulinier, 1907. 16mo. 64 pp. Fr. 0.10

Fuoini (C.). Algebra elementare per gl'istituti nautici. 3a edizione. Genova, Gioventù, 1908. 8vo. 252 pp.

——. Geometria piana e nozioni di geometria solida, per le scuole secondarie inferiori. 8a edizione. Genova, Gioventù, 1908. 8vo. 143 pp. L. 1.50

Galli (F.). Elementi di algebra per le scuole medie di primo grado. Cremona, Fezzi, 1907. 8vo. 58 pp. L. 0.75

Ghiotti (E.). Nozioni pratiche di aritmetica e geometria per le scuole operaie, professionali e tecniche. Nuova edizione. Torino, Gallizio, 1908. 16mo. 6 + 243 pp.

Gravelaar (N. L. W.). Leerboek der planimetrie. 3er druk. Groningen, 1907. 8vo. 192 pp. M. 2.50

Grévy (A.). Eléments d'algèbre à l'usage des élèves des classes de troisième A et première A et B (programmes de 1905). 6e édition. Paris, Vuibert, 1907. 16mo. 232 pp. Fr. 1.75

——. Géométrie théorique et pratique, conforme au programme du 27 juillet 1905, à l'usage des élèves du premier cycle B (classes de cinquième B à troisième B). 4e édition. Paris, Vuibert, 1908. 16mo. 7 + 473 pp. Fr. 3.50

——. Trigonométrie à l'usage des élèves des classes de première C et D et de mathématiques A et B (programmes de 1905). 4e édition. Paris, Vuibert, 1908. 16mo. 280 pp. Fr. 2.25

Guichard (C.). Traité de géométrie. Vol. I, à l'usage des classes de seconde et première C et D et mathématiques A et B. 3e édition, conforme au programme du 27 juillet, 1905. Paris, Vuibert, 1908. 8vo. 8 + 568 pp. Fr. 6.00

Jacobsthal (W.). See Weber (H.).

Jahne (J.) und Barbisch (H.). Leitfaden der Geometrie und des geometrischen Zeichnens für Knabenbürgerschulen. Ausgegeben in 1 Bande. 2te, verbesserte Auflage. Wien, Manz, 1907. 8vo. 8 + 185 pp. M. 2.10.

Keller (S. S.). Algebra. New York, Van Nostrand, 1907. 12mo. 113 pp. Cloth. $1.00

——. Plane and solid geometry. New York, Van Nostrand, 1907. 12mo. 212 pp. Cloth. $1.50

Mainardi (M.). Una lezione di trigonometria. Napoli, 1907. 8vo. 8 pp.

Matriculation model answers. Mechanics. Sept. 1903–Sept. 1907. New edition. London, Clive, 1907. 8vo. 97 pp. 2s.

Mayer (J. E.). Mathematik für Techniker. Gemeinverständliches Lehrbuch der Mathematik für Mittelschüler sowie besonders für den Selbstunterricht. Vol. 5. Leipzig, Schäfer, 1907. 8vo. 8 + 180 pp. M. 2.40

Rudio (F.). Des Simplicius Bericht über die Quadraturen des Antiphon und des Hippokrates, griechisch und deutsch. Mit einem historischen Erläuterungsberichte als Einleitung. Im Anhang ergänzende Urkunden, verbunden durch eine Uebersicht über die Geschichte des Problems von der Kreisquadratur vor Euklid. Leipzig, Teubner, 1907. 8vo. 10 + 184 pp. (Urkunden zur Geschichte der Mathematik im Altertume, 1es Heft.) M. 4.80

Rüefli (J.). Lehrbuch der ebenen Geometrie nebst einer Sammlung von Uebungsaufgaben. Zum Gebrauche an Sekundarschulen (Realschulen) und Gymnasial-Anstalten, sowie zum Selbststudium bearbeitet. 4te Auflage. Bern, Francke, 1907. 8vo. 8 + 199 pp. M. 2.40

Salmoiraghi (A.). See Bremiker (C.).

Schmidt (A.). Beiträge zum mathematischen Unterricht: Berechnung der Logarithmen in Untersekunda; Einführung der komplexen Zahlen. Berlin, 1907. 4to. 16 pp. M. 1.00

Schubert (H.). Mathematische Mussestunden. Eine Sammlung von Geduldspielen, Kunststücken und Unterhaltungsaufgaben mathematischer Natur. Kleine Ausgabe. 3te Auflage. Leipzig, Göschen, 1907. 8vo. 306 pp. Cloth. M. 5.00

Vallín y Bustillo (A. F.). Elementos de matemáticas. Obra declarada de texto por los gobiernos de España y Portugal. Nueva edición estereotípica. Madrid, Sucesores de Hernando, 1907. 226 pp. P. 7.00

Vásquez Queipo (V.). Tablas de los logaritmos vulgares de los numeros 1–20000 y de las lineas trigonometricas. 24a edición. Madrid, 1907. 8vo. 268 pp. P. 5.00

Vives y Casademont (L.). Elementos de algebra. Malaga, "La Española," 1907. 240 pp.

Weber (H.) und Wellstein (J.). Encyklopädie der Elementar-Mathematik. Ein Handbuch für Lehrer und Studierende. (In 3 Bänden.) Vol. II: Encyklopädie der elementaren Geometrie. Bearbeitet von H. Weber, J. Wellstein und W. Jacobsthal. 2te Auflage. Leipzig, Teubner, 1907. 8vo. 12 + 596 pp. Cloth. M. 12 00

Wellstein (J.). See Weber (H.).

Westlake (H. F.). Construction in practical geometry. London, Philip, 1907. 8vo. 62 pp. 1s.

Wheeler (A. H.). Algebra for grammar schools, with mental exercises. Boston, Little, 1907. 12mo. 12 + 186 pp. Cloth. $0.50

——. First course in algebra; with 8000 examples, including 3000 mental exercises. Boston, Little, 1907. 12mo. 19 + 664 pp. Cloth. $1.15

III. APPLIED MATHEMATICS.

CHOLLET (T.) et MINEUR (P.). Traité de géométrie descriptive. 1re partie à l'usage des élèves des classes de première C et D. 5e édition, conforme au programme du 27 juillet 1905. Paris, Vuibert, 1908. 8vo. 8 + 336 pp. Fr. 3.50

CONTALDI (P.). La meccanica nella scuola e nell' industria. Vol. I: Meccanica generale; meccanica applicata. 2a edizione riveduta ed ampliata. Milano, Hoepli, 1908. 8vo. 15 + 711 pp. L. 16.00

FORNER CARRATALÁ (F.). Manual de mecánica. Mecánica elemental. 2 tomos. Barcelona, 1907. 8vo. 246 and 205 pp. P. 5.00

GARNERI (A.). Corso elementare di disegno geometrico. Parte I: Problemi grafici e geometrici e ornamentazione geometrica. Fasc. 1. Torino, Paravia, 1908. 16mo. 48 pp.

GUICHARD (C.). Traité de mécanique à l'usage des élèves de mathématiques A et B et des candidats aux écoles. 5e édition, conforme au programme du 27 juillet 1905. Paris, Vuibert, 1908. 8vo. 7 + 248 pp.

HERRMANN (G.). See WEISBACH (J.).

LOVE (A. E. H.). Theoretical mechanics: an introductory treatise on the principles of dynamics, with applications and numerous examples. 2d edition. New York, Putnam, 1907. 8vo. 16 + 367 pp. Cloth. $3.00

MAURER (E. R.). Strength of materials: a practical manual of scientific methods of locating and determining stresses and calculating the required strength and dimensions of building materials. Chicago, American School of Correspondence, 1908. 8vo. 3 + 128 pp. Cloth. $1.00

MINEUR (P.). See CHOLLET (T.).

PEDROTTI (L.). Importanti applicazioni dei logaritmi, funzioni esponenziali e calcolo degli interessi composti ed annualità. Milano, Società ed. Sonzogno, 1907. 16mo. 63 pp. L. 0.15

POPPLEWELL (W. C.). Strength of materials: a manual for students of engineering. New York, Van Nostrand, 1907. 8vo. 11 + 180 pp. Cloth. $2.00

RUIZ-CASTIZO (J.). Tratado de mecánica racional, apropiado á la enseñanza en las facultades de ciencias y en las escuelas especiales. 2o fasc.: Cinemática. Madrid, Suarez, 1907. Pp. 305–589. P. 9.00

SCHÄRTLIN (G. G.). Zur mathematischen Theorie der Invaliditätsversicherung. Bern, 1907. 4to. 43 pp. M. 2.00

SCHULTZ (E.). Mathematische und technische Tabellen für Maschinenbauschulen und für den Gebrauch in der Praxis. Ausgabe A, mit hinten lose eingehängter Anleitung. 7te Auflage. Essen, Baedeker, 1908. 8vo. 4 + 228 pp. M. 2.00

SOTHERN (J. W. and R. M.). Simple problems on marine engineering design. Glasgow, Munro, 1907. 8vo. 200 pp. 2s. 6d.

WEISBACH (J.) and HERRMANN (G.) The mechanics of hoisting machinery. New edition. London, Macmillan, 1907. 8vo. Cloth. 12s. 6d.

WHITTAKER (E. T.). The theory of optical instruments. (Cambridge tracts in mathematics and mathematical physics; No. 7.) Cambridge, University Press, 1907. 8vo. 80 pp. 2s. 6d.

ZEUNER (G.). Technical thermodynamics. 2 vols. First English edition. London, Constable, 1907. 8vo. 472 and 564 pp. Cloth. 36s.

LIBRARY

SUBJECTIVE GEOMETRY.

BY DR. G. W. HILL.

(Read before the American Mathematical Society, February 29, 1908.)

THE fact that many persons suppose that the non-euclidean and euclidean geometries contradict each other seems to call for further elucidation of the matter. Things that disagree do not necessarily contradict each other. One circumstance that leads to the prevalent obscurity in this matter is the use of diagrams and objective phraseology in the treatment of synthetic geometry. The reader always supposes that geometric figures are something he can see and handle. Also, in Euclid's mode of presenting the science, it is difficult to see what is fundamental and what derivative.

It is said that Legendre cast about for a long time to find the proof of the theorem "The sum of the three angles of a triangle is equivalent to two right angles" without success. He should have seen that his failure arose from the want of due determination in his definition. The theorem is true if the proper concepts are attached to the words "triangle," "angle" and "right angle."

Let us illustrate by an instance from algebra. The man who knows nothing about negative and complex quantity will say "No cubic equation can have more than three roots, but there are some which have less." The man who admits these sorts of quantity says "Every cubic equation has three roots." These statements disagree, but are not in contradiction. The case is the same as if they were written in different languages. We can translate the first statement so as to agree with the second by simply writing "positive real root" instead of "root."

The difference between the non-euclidean and the euclidean geometry is one of mere definition of terms. We shall see this more easily if we treat geometry as a subjective science. There are two methods for its exposition called synthetic and analytic. The latter makes use of the symbolic notation of algebra, and is more easily comprehended if the notion of a linear unit is admitted. At the end of the investigation, however, the latter can be brushed away as mere scaffolding, and what is left is magnitudinal geometry in place of mensuration.

We shall employ the analytic method. First it is necessary to understand what is meant by a manifold. Let x be a variable capable of taking all real values, and let these values be arranged in the order of their relative magnitudes. This is the manifold of one dimension. We may objectively illustrate it by the graph

$$\infty, \cdots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5 \cdots, \infty,$$

exhibiting the two series of positive and negative integers separated by 0. The symbol ∞ here simply denotes unlimited extension. The values x may take are called the elements of the manifold. Again we may suppose another variable y, independent of x but able to take a series of values in the same way. Now let this manifold of one dimension exist in conjunction with each one of the elements of the manifold of one dimension for x; that is, there is constituted a sort of table of double entry. This is the manifold of two dimensions in reference to the two variables x and y. The element of the second manifold is formed by joining any value of x to any value of y. The manifold of two dimensions may be illustrated thus (the square of heavy dots denotes contemplated elements, the light dots indicated elements, and the whole is bordered with the symbol ∞):

$$\begin{array}{ccccccccc}
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \infty \\
\infty & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \infty \\
\infty & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \infty \\
\infty & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \infty \\
\infty & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \infty \\
\infty & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \infty \\
\infty & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty
\end{array}$$

Proceeding in like manner, and adopting a third variable z, we can form the manifold of three dimensions; and so on.

Let us use the letter M for manifold. Thus the three manifolds we have considered are designated by the symbols

$$M(x), \qquad M(x, y), \qquad M(x, y, z).$$

Now let us suppose that a, b, c denote three constant quantities, and let the first be added to every value of x, the second to every value of y, and the third to every value of z. Then it is evident we may write the identities

$$M(x + a) \equiv M(x), \quad M(x + a, y + b) \equiv M(x, y),$$
$$M(x + a, y + b, z + c) \equiv M(x, y, z).$$

The property of a manifold shown by these identities may be called the isotropy of transposition. It is important to bear in mind that we have established this property only by insisting that the boundary of the manifold is indefinitely removed.

We must here explain some metaphysical terms. The attributes of an entity are whatever may be rightfully considered as belonging or attached to it. We divide attributes into the two classes of properties and characteristics. Properties are attributes which cannot be ignored, they are essential to the constitution of the entity. But characteristics admit of choice. To apply this: a manifold has but one property, viz., that of the isotropy of transposition.

We can now consider the connection of the manifold of any number of dimensions with geometry of the same number of dimensions. Geometry is the science of space. What is space? It is the continuum which holds the manifold. Is anything contained in the continuum besides the manifold? If it is subjective geometry we are dealing with, the answer is nothing. Then the attributes of space are identical with those of the manifold. In our reasonings we may everywhere substitute the word manifold for the word space.

To make an application of what we have discovered: It has been questioned whether Euclid was warranted in employing his method of proof by superposition. There need not be any question about this, for the isotropy of transposition justifies it.

We proceed now to investigate possible characteristics for the manifold. Since the manifold is unique for a given number of dimensions, it follows that the number of independent characteristics, necessary for thoroughly defining the manifold, is equal to the number of its dimensions. We can build up the system of characteristics by starting from the manifold of one dimension. In a manifold of n dimensions, if we suppose one of the variables to be limited to a single value, and, as another step, the variable to be disregarded, the result is a manifold of $n - 1$ dimensions. Then, in selecting the characteristics for our manifold, we can suppose the characteristics of the manifold of two dimensions are congruent with those of the manifold of one dimension. And, again, that the characteristics of the manifold of three dimensions are congruent with those of two dimensions, and so on. For the sole function of geometry is to ascertain the relativities of special elements of the manifold. We therefore start with the manifold of one dimension.

Let the special values of the variable x be denoted by x' and x''. Here the only possible relativities are two, viz., $x'' - x'$ and $x' - x''$. But we insist that these relativities shall be divided so that one is always positive and the other shall express the idea of advance or recession. Then, calling the first distance, and denoting it by D, we might write the equation $D = \pm (x'' - x')$. But, not liking the ambiguous sign, we prefer to write

$$D = \sqrt{(x'' - x')^2},$$

where the radical must be taken positively. Then, similarly, for the manifold of one dimension, whose variable is y, we have

$$D = \sqrt{(y'' - y')^2},$$

and, again, for one whose variable is z,

$$D = \sqrt{(z'' - z')^2}.$$

How can D, for two dimensions, be made congruent with the D of one dimension? And, again, the D of three dimensions congruent with the D of two dimensions? It is plain that this will be accomplished in the simplest manner, if, in the first case, we put

$$D = \sqrt{(x'' - x')^2 + (y'' - y')^2},$$

and, in the second case,

$$D = \sqrt{(x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2}.$$

For higher dimensions we shall have similar expressions. It must be borne in mind that these things are matters for choice.

What we have done suffices for the manifold of one dimension. But, for the manifold of two dimensions, we must have an additional characteristic. We call this λ, and choose to put

$$\frac{x''-x'}{D}=F_1(\lambda),\quad \frac{y''-y'}{D}=F_2(\lambda),$$

where F_1 and F_2 denote two functions not yet determined. We derive from these equations

$$[F_1(\lambda)]^2+[F_2(\lambda)]^2=1,$$

and

$$F_1(\lambda)=\pm\sqrt{1-[F_2(\lambda)]^2}.$$

Thus it is plain that the corresponding values of $F_1(\lambda)$ and $F_2(\lambda)$ can be tabulated in a table to single entry, that F_1 and F_2 are both periodic functions of λ to the same, but arbitrary, period, and that the phase of F_2 is a quarter of a period behind the phase of F_1. Consequently, we may put

$$F_1(\lambda)=\cos(m\lambda+\alpha),\quad F_2(\lambda)=\sin(m\lambda+\alpha),$$

where m and α are arbitrary constants. The latter are at our disposal, only m must not, for efficiency's sake, receive either of the values 0, ∞. Thus, for the sake of simplicity, we put $m=1$, $\alpha=0$. This comes to the same thing as saying that, what we first called $m\lambda+\alpha$, we now designate as λ. It must be remembered that we have put no restrictions on λ, it simply has the capacity of variation. Now we have

$$F_1(\lambda)=\cos\lambda,\quad F_2(\lambda)=\sin\lambda,$$

and the period of the two functions is 2π.

In the case of three dimensions, adopting a third variable θ, we choose to put

$$\frac{x''-x'}{D}=F_1(\lambda,\theta),\quad \frac{y''-y'}{D}=F_2(\lambda,\theta),\quad \frac{z''-z'}{D}=F_3(\lambda,\theta),$$

whence is derived

$$[F_1(\lambda,\theta)]^2+[F_2(\lambda,\theta)]^2+[F_3(\lambda,\theta)]^2=1.$$

In attempting the solution of this functional equation, we note that it is not necessary for our purposes that we should arrive at a general solution, but only at any particular solution that is efficient. Thus we readily see that

$$F_1(\lambda, \theta) = \cos(m'\theta + \beta)\cos(m\lambda + \alpha),$$
$$F_2(\lambda, \theta) = \cos(m'\theta + \beta)\sin(m\lambda + \alpha),$$
$$F_3(\lambda, \theta) = \sin(m'\theta + \beta),$$

where m, m', α, β are arbitrary constants, which may have values adopted at will, except that m and m' must not receive either of the values 0, ∞. For like reasons as before, we assume $m = m' = 1$, $\alpha = \beta = 0$. Thus we have

$$\frac{x'' - x'}{D} = \cos\theta\cos\lambda, \quad \frac{y'' - y'}{D} = \cos\theta\sin\lambda, \quad \frac{z'' - z'}{D} = \sin\theta.$$

These formulas are purposely made to be congruent with those for two dimensions; if we make $\theta = 0$, and then disregard it and z, the formulas reduce to those for two dimensions.

It is not deemed necessary to notice the formulas for higher dimensions than three; they can be found in Jacobi, Vorlesungen über Dynamik, page 185.

We must notice the limits between which the new variables D, λ, θ play. While the old variables x, y, z play each between $-\infty$ and $+\infty$, D plays between 0 and $+\infty$, λ between 0 and 2π and θ between $-\pi/2$ and $+\pi/2$. It is not forbidden to extend λ and θ beyond these limits, but it must be borne in mind that the consequence of doing this is to repeat the field.

After this exposition it is easy to point out what distinguishes non-euclidean from euclidean geometry. In the first place the non-euclidean geometer insists that there may be something else in the continuum, which space is, beside the manifold; but does not define what it is. Of course it is = objective geometry — subjective geometry. In the second place he conceives that this vague something takes away from him the liberty of putting $m = 1$ in the argument $m\lambda + \alpha$. Accordingly he carries through all his ratiocinations in geometry allowing m to stand indeterminate. This course introduces no error into his science any more than the introduction of $i = \sqrt{-1}$ into arithmetic and algebra made those sciences erroneous. But it ought to be recognized as an innovation, to be justified by the advantages gained. With regard to $\sqrt{-1}$ it is almost the universal opinion of mathematicians that it is an improvement. But the approval of non-euclidean geometry is not so general; this may be due to its not having been so long before the public. The gist of the argument in favor of this view of the science is that it is not necessary to fix the period of the variable expressing orientation, that is the value of m. But both classes of geometers put $a = 0$; why should it be prohibited to put $m = 1$?*

As further illustrating the subject, let us consider the theorem Legendre wished to establish. We must begin by a definition. A triangle is the bundle of relativities of three elements of the manifold of two dimensions. Let these elements be denoted as

$$x', y' \text{ and } x'', y'' \text{ and } x''', y'''.$$

Then we have the equations

$$(x'' - x') + (x''' - x'') + (x' - x''') = 0,$$
$$(y'' - y') + (y''' - y'') + (y' - y''') = 0.$$

We may adopt D, λ and D', λ' and D'', λ'', such that

$$x'' - x' = D \cos \lambda, \quad x''' - x'' = D' \cos \lambda', \quad x' - x''' = D'' \cos \lambda'',$$
$$y'' - y' = D \sin \lambda, \quad y''' - y'' = D' \sin \lambda', \quad y' - y''' = D'' \sin \lambda'',$$

where the λ are taken between the limits 0 and 2π. Consequently

$$D \cos \lambda + D' \cos \lambda' + D'' \cos \lambda'' = 0,$$
$$D \sin \lambda + D' \sin \lambda' + D'' \sin \lambda'' = 0.$$

From these equations may be derived the value of each of the distances expressed in terms of the others and the λ's. The equations determining these values can be written

$$D^2 = D''^2 - 2D''D' \cos [\pm \pi \pm (\lambda'' - \lambda')] + D'^2,$$
$$D'^2 = D^2 - 2DD'' \cos [\pm \pi \pm (\lambda - \lambda'')] + D''^2,$$
$$D''^2 = D'^2 - 2D'D \cos [\pm \pi \pm (\lambda' - \lambda)] + D^2.$$

The ambiguous signs in the arguments of the cosines are to be so taken that each argument may result between the limits 0 and π. As there are six ambiguous signs, there are no less than 64 different combinations to be selected from. By a few simple considerations we get rid of the superfluous ones. We can denote the arguments by A, B, C, A belonging to D, B to D' and C to D''. Thence the simpler forms

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* It is a curious psychological commentary on this matter that the astronomer, who has a personal equation in measuring angles of position, does unconsciously what both classes of geometers consciously refrain from doing.

$$D^2 = D''^2 - 2D''D' \cos A + D'^2,$$
$$D'^2 = D^2 - 2DD'' \cos B + D''^2,$$
$$D''^2 = D'^2 - 2D'D \cos C + D^2.$$

A, B, and C are the angles of the triangle. From the circumstance that in writing the three arguments we have used a cyclical permutation of the symbols λ, λ', λ'', it follows that we are not permitted to choose at random either of the components of the second ambiguous sign, but if we take the upper in any one case we must so take it in all, and similarly with respect to the lower. This is no more than saying that $\lambda'' - \lambda'$ belongs exclusively to D, $\lambda - \lambda''$ to D' and $\lambda' - \lambda$ to D''. Thus we may write the two cases for the expressions of the arguments

$$\pm \pi + (\lambda'' - \lambda'), \ \pm \pi + (\lambda - \lambda''), \ \pm \pi + (\lambda' - \lambda),$$

and

$$\pm \pi - (\lambda'' - \lambda'), \ \pm \pi - (\lambda - \lambda''), \ \pm \pi - (\lambda' - \lambda).$$

We may now inquire what is the sum of the three arguments or angles. Thus

$$A + B + C = \pm \pi \pm \pi \pm \pi.$$

The condition that each angle is contained between the limits 0 and π makes the sum either π or 3π. But, in order that the latter value may hold, it is necessary that $A = B = C = \pi$; this is inadmissable for the reason that some triangles exist in which one of the angles is less than π, and this condition of things must be continuous with the former state. Thus the sum of the angles would move from 3π towards π which is contrary to the equation. The value 3π must be rejected, and we have

$$A + B + C = \pi,$$

which is what Legendre was desirous of proving.

Euclid's eleventh axiom is a clumsy way of rectifying the inadequate definition of what he meant by an angle; it is clearly introduced as an afterthought. This led Legendre and many others to think that this axiom was superfluous and to attempt to construct geometries in which it was not employed. If an angle is described as a something that is attached to the vertex of a triangle, although it is shown how angles may be added, multiplied or divided, this proves nothing as to the actual magnitude of angles. The angles of a spherical triangle are also attached to the vertices and can be added, multiplied or divided in a similar way, but their actual magnitudes are different from the similar angles in a space of two dimensions. Attention ought to be directed to the fact that saying the angle is included between the sides of the triangle which come together and form the vertex adds nothing to the determination of the term "angle."

As a last point in criticism, many authors on geometry introduce misconceptions and misapprehensions into the matter by employing the concepts of the next higher dimension in a discussion which ought to be exclusively in terms belonging to the dimension dealt with. The student should be warned that lines, either straight or curved, have no existence in a space of one dimension, and that planes and curved surfaces do not exist in a space of two dimensions. This would put an end to the talk about a geometry of two dimensions on the surface of a sphere.

ON HIGHER CONGRUENCES AND MODULAR INVARIANTS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, February 29, 1908.)

1. THE object of this paper is to give a two-fold generalization of Hurwitz's* explicit formula for the number of integral roots of a given congruence modulo p, p being prime. On the one hand, we may derive an equally simple formula which gives, apart from a multiple of p, the number of the roots of a specified order ($\leqq t$) of irrationality; viz., the roots belonging to the Galois field of order p^t. On the other hand, the problem may, without loss of simplicity, be further generalized†

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* *Archiv der Math. u. Physik* (3), vol. 5 (1903), p. 17.

† Other generalizations of theoretical importance, but not leading readily to explicit expressions in terms of the coefficients, are given by H. Kühne, *Archiv der Math. u. Physik*, vol. 6 (1904), p. 174.

by replacing the initial field of integers modulo p by an arbitrary Galois field, $GF[p^n]$.

From his formula Hurwitz deduces an (absolute) invariant* of degree $p-1$ of the general binary form under linear transformations taken modulo p. He states that one of the fundamental questions in the theory of modular invariants is "die Frage der Endlichkeit": whether or not all the invariants can be expressed as rational integral functions with integral coefficients of a finite number of the invariants. Emphasizing the difficulty of this question, he answers it only for a special case. As a matter of fact, this question is trivial for modular invariants, since any polynomial in $a_1, \cdots, a_r$ with integral coefficients is congruent modulo p to a polynomial in which the exponent of each a_i is at most $p-1$ (in view of Fermat's theorem $a^p \equiv a$).

2. Consider an equation, with coefficients in any given Galois field $GF[p^n]$ of order p^n,

$$(1) \qquad f(x) \equiv a_0 + a_1 x + \cdots + a_r x^r = 0.$$

Let N denote the number of its roots, different from zero, which belong to the $GF[p^{nm}]$. For brevity, set $P = p^{nm}$. Then N is the number of the vanishing expressions $f(\xi)$, where ξ ranges over the $P-1$ marks $\neq 0$ of the $GF[P]$. In that field,

$$1 - k^{P-1} = \begin{cases} 1 \text{ if } k = 0, \\ 0 \text{ if } k \neq 0. \end{cases}$$

Hence $N \equiv N^*$ (mod p), where

$$N^* = \sum_{\xi} \{1 - [f(\xi)]^{P-1}\} \equiv -1 - \sum_{\xi} [f(\xi)]^{P-1}.$$

Employing the algebraic expansion

$$(2) \qquad [f(\xi)]^{P-1} = \sum_{\mu} C_\mu \xi^\mu,$$

and summing for the marks $\xi \neq 0$ of the $GF[P]$, we get

$$N^* = -1 - \sum_{\mu} C_\mu (\sum_{\xi} \xi^\mu).$$

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* For a cubic form this invariant is exhibited in expanded form for $p \leqq 13$ on p. 221 of my paper on modular invariants, *Trans. Amer. Math. Soc.*, vol. 8 (1907), pp. 205–230. At the time of writing that paper I did not know of Hurwitz's work.

By the writer's Linear Groups, page 54,

$$\sum_{\xi} \xi^{\mu} = \begin{cases} -1 \text{ for } \mu \text{ divisible by } P-1, \\ 0 \text{ for } \mu \text{ not divisible by } P-1. \end{cases}$$

Hence

$$N^* + 1 = \sum C_{\nu(P-1)} \qquad (\nu = 0, 1, \cdots). \tag{3}$$

Inserting the values of P and the C's, we get

$$N^* + 1 = \sum_{\alpha} \frac{(p^{nm}-1)!}{\alpha_0!\, \alpha_1! \cdots \alpha_r!} a_0^{\alpha_0} a_1^{\alpha_1} \cdots a_r^{\alpha_r}, \tag{4}$$

summed for all sets of integers $\alpha_i \geqq 0$ for which

$$\alpha_0 + \alpha_1 + \cdots + \alpha_r = p^{nm} - 1, \tag{5}$$

$$\alpha_1 + 2\alpha_2 + \cdots + r\alpha_r \equiv 0 \pmod{p^{nm} - 1}. \tag{6}$$

In (4) the fractional form of the multinomial coefficient is to be replaced by its integral value. In fact, certain of the α's may be multiples of p[*].

3. When $nm > 1$, certain of the multinomial coefficients in (4) are multiples of p and the corresponding terms may be dropped from the equation. To this end, set

$$\alpha_i = \sum_{j=0}^{mn-1} c_{ij} p^j \qquad (0 \leqq c_{ij} < p), \tag{7}$$

for $i = 0, 1, \cdots, r$. Further,

$$p^{nm} - 1 = \sum_{j=0}^{mn-1} (p-1) p^j.$$

Then, by well-known theorems, the multinomial coefficient M in (4) is a multiple of p unless

$$\sum_{i=0}^{r} c_{ij} = p - 1 \quad (j = 0, 1, \cdots, mn - 1); \tag{5'}$$

while, if these relations hold,

$$M \equiv \prod_{j=0}^{mn-1} \frac{(p-1)!}{c_{0j}!\, c_{1j}! \cdots c_{rj}!} \pmod{p}. \tag{8}$$

In place of (5) we may employ the more exacting relations (5').

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[*] As this is not true in Hurwitz's case $n = m = 1$, we may then place the factor $(p-1)!$ before the summation sign.

Finally, the exponents of each a_i may be made less than p^n by employing

$$(9) \qquad a_i^{p^n} = a_i.$$

4. Instead of deleting vanishing terms of (4) by employing relations (5′) in place of (5), we may modify our former method of determining the C_μ. Instead of the algebraic expansion (2) we may employ the modular relation

$$(10) \quad [f(\xi)]^{p^{nm}-1} \equiv \prod_{j=0}^{m-1} [f(\xi)]^{p^{nj}(p^n-1)} = \prod_{j=0}^{m-1} [f(\xi^{p^{nj}})]^{p^n-1},$$

which follows readily from (9).

The plan of § 3, which employs (9) only at the final stage, has the advantage of furnishing simultaneously the number of roots in the $GF[p^t]$ of equation (1) with coefficients in any $GF[p^n]$, where n is a divisor of t. See the example in § 6.

5. In (1) set $x = x_2/x_1$ and consider the binary form

$$(11) \qquad f(x_1, x_2) = a_0 x_1^r + a_1 x_1^{r-1} x_2 + \cdots + a_r x_2^r$$

in the $GF[p^n]$. Two pairs of marks (x_1, x_2) and (x_1', x_2') of the $GF[p^{nm}]$ will be said to give the same set of solutions of

$$(12) \qquad f(x_1, x_2) = 0$$

if, and only if, there exists a mark ρ of the $GF[p^{nm}]$ for which $x_1' = \rho x_1$, $x_2' = \rho x_2$ in that field. Let A denote the number of distinct sets of solutions, not both zero, of (12) in the $GF[p^{nm}]$. Evidently

$A = N$, when $a_0 \neq 0$ and $a_r \neq 0$ in the $GF[p^n]$;

$A = N + 1$, when just one of the coefficients a_0, a_r is zero;

$A = N + 2$, when $a_0 = a_r = 0$.

Hence in every case $A \equiv A^*$ (mod p), where

$$A^* + a_0^{p^n-1} + a_r^{p^n-1} = N + 2.$$

THEOREM. *The number A of distinct sets of solutions, not both zero, in the $GF[p^{nm}]$ of a homogeneous equation* (12), *with coefficients in the $GF[p^n]$, is congruent to A^* modulo p, where*

$$(13) \quad A^* - 1 = -a_0^{p^n-1} - a_r^{p^n-1} + \sum_{\alpha} \frac{(p^{nm}-1)!}{\alpha_0!\,\alpha_1!\cdots\alpha_r!} a_0^{\alpha_0} a_1^{\alpha_1} \cdots a_r^{\alpha_r},$$

subject to (5) *and* (6), *or to* (6), (7) *and* (5′).

The function defined by the second member of (13) *is an absolute modular invariant of the binary form* (11).

6. Examples. Let $r = 3$, so that (11) is a binary cubic. First, let $p = 3$, $m = n = 1$; the invariant* is seen to be

$$K = a_1^2 + a_2^2 - a_0a_2 - a_1a_3. \tag{14}$$

Next, let $p = 3$, $nm = 2$. As in (7), set

$$a_i = c_{i0} + 3c_{i1} \qquad (c_{ij} = 0, 1, 2).$$

Then, by (5′) and (6),

$$\sum_{i=0}^{3} c_{i0} = 2,\ \sum_{i=0}^{3} c_{i1} = 2,\ c_{10} + 2c_{20} + 3c_{30} + 3c_{11} + 6c_{21} + 9c_{31} \equiv 0 \pmod 8).$$

The sets of values of the c_{ij} are as follows:

c_{01}	c_{11}	c_{21}	c_{31}	c_{00}	c_{10}	c_{20}	c_{30}
2	0	0	0	2	0	0	0
1	0	1	0	1	0	1	0
1	1	0	0	0	0	1	1
1	0	1	0	0	2	0	0
0	0	1	1	1	1	0	0
0	2	0	0	1	0	1	0
0	0	2	0	0	0	2	0
0	1	0	1	0	0	2	0
0	0	2	0	0	1	0	1
0	0	0	2	0	0	0	2
0	1	0	1	0	1	0	1
0	2	0	0	0	2	0	0

In view of (8), the resulting function (13) is

$$\begin{aligned} a_0^4a_2^4 + a_0^3a_1^3a_2a_3 - a_0^3a_1^2a_2^3 + a_0a_1a_2^3a_3^3 - a_0a_1^6a_2 \\ + a_2^8 - a_1^3a_2^2a_3^3 - a_1a_2^6a_3 + a_1^4a_3^4 + a_1^8. \end{aligned} \tag{15}$$

Taking $n = 2$, $m = 1$, we have in (15) an absolute invariant† of the binary cubic in the $GF[3^2]$.

Taking $n = 1$, $m = 2$, and reducing by $a^3 \equiv a$, we get

$$\begin{aligned} J = a_0^2a_2^2 - a_0a_1a_2a_3 + a_0a_1^2a_2 + a_2^2 \\ + a_1a_2^2a_3 + a_1^2a_3^2 + a_1^2 \pmod 3), \end{aligned} \tag{15'}$$

an absolute invariant of the binary cubic in the $GF[3]$.

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* See (14), *Transactions*, l. c., p. 211.

† See (78), *Transactions*, l. c., p. 227; direct by (58), p. 222.

Now K and J, increased by unity, give (apart from a multiple of 3) the number of sets of values for which a cubic form (with the coefficients not all zero) vanishes in the $GF[3]$ and the $GF[3^2]$, respectively. We find that *

$$J = K + \Delta^2 - \Delta \quad (\Delta = \text{discriminant}),$$

$$K^2 + K = J^2 + J.$$

But K is not a rational function of J (in view of the first and second forms below), nor J a rational function of K (in view of the second and third forms):

Form.	K	J.	Δ
$x^3 - xy^2 + y^3$	-1	-1	1
$x^3 + xy^2$	0	-1	-1
x^3	0	0	0
$x^2y + xy^2$	-1	-1	1
x^2y	1	1	0
Vanishing	0	0	0

Every cubic can be transformed modulo 3 into one of those given in the table (*Transactions*, l. c., page 232).

THE UNIVERSITY OF CHICAGO,
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NOTE ON JACOBI'S EQUATION IN THE CALCULUS OF VARIATIONS.

BY PROFESSOR MAX MASON.

(Read before the American Mathematical Society, February 29, 1908.)

IN Weierstrass's theory of the calculus of variations † it is shown that the determinant

$$\omega = \frac{\partial y}{\partial t}\frac{\partial x}{\partial a} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial a}$$

formed from the equations $x = x(t, a)$, $y = y(t, a)$ of a family of extremals of the integral

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* If we employ the invariant $P = \Delta + 1 - K$ (l. c., p. 211), we have

$$J = K^2 + K + P - 1.$$

† See for example Bolza, Lectures on the calculus of variations, Chicago, 1904.

$$J = \int F(x, y, x', y')dt$$

is a solution of Jacobi's equation

$$(\omega' F_1)' - \omega F_2 = 0.$$

This result, which is of fundamental importance in the theory, is obtained by differentiating the Euler equations of the extremals

$$F_x - \frac{d}{dt} F_{x'} = 0, \quad F_y - \frac{d}{dt} F_{y'} = 0$$

with respect to the parameter a, a method which involves considerable reckoning and the introduction of two sets of functions $L, M, N; L_1, M_1, N_1$, which serve to define F_2.

It is the object of this note to derive the result above stated directly from the single equation of the extremals

$$T \equiv F_{xy'} - F_{yx'} + F_1(x'y'' - y'x'') = 0, \tag{1}$$

which is equivalent to the pair of dependent Euler equations. The introduction of successive sets of auxiliary functions to define F_2 is in this way avoided, and an explicit form for F_2 is obtained.

Write for abbreviation $\partial x/\partial a = \xi$, $\partial y/\partial a = \eta$, and denote differentiation with respect to t by accents. Then

$$\omega = y'\xi - x'\eta, \quad \omega' = y''\xi - x''\eta + y'\xi' - x'\eta',$$
$$\omega'' = y'''\xi - x'''\eta + 2(y''\xi' - x''\eta') + y'\xi'' - x'\eta''.$$

If equation (1) be differentiated with respect to a, and the quantity $[y'''\xi - x'''\eta + 3(y''\xi' - x''\eta')]F_1$ be subtracted and added in the result, the following equation is obtained:

$$\begin{aligned} &- \omega'' F_1 + \xi'[3y''F_1 + (x'y'' - y'x'')F_{1x'} - x'y'F_{1x} - y'^2 F_{1y}] \\ &\quad + \eta'[- 3x''F_1 + (x'y'' - y'x'')F_{1y'} + x'^2 F_{1x} + x'y'F_{1y}] \\ &\quad + \xi[y'''F_1 + (x'y'' - y'x'')F_{1x} + F_{xxy'} - F_{xyx'}] \\ &\quad + \eta[- x'''F_1 + (x'y'' - y'x'')F_{1y} + F_{xyy'} - F_{yyx'}] = 0. \end{aligned} \tag{2}$$

Since

$$F_1' = x''F_{1x'} + y''F_{1y'} + x'F_{1x} + y'F_{1y},$$

the coefficients of ξ' and η' are equal to

$$y''(3F_1 + x'F_{1x'} + y'F_{1y'}) - y'F_1',$$
$$-x''(3F_1 + x'F_{1x'} + y'F_{1y'}) + x'F_1',$$

respectively. Now it may be shown from the homogeneity property of F that

$$(3) \qquad 3F_1 + x'F_{1x'} + y'F_{1y'} = 0.$$

In fact, on differentiating the identity

$$(4) \qquad x'F_{x'} + y'F_{y'} = F$$

twice with respect to x', the equation

$$F_{x'x'} + x'F_{x'x'x'} + y'F_{x'x'y'} = 0$$

is obtained. If the second derivatives be expressed in terms of F_1, this equation becomes

$$y'^2(3F_1 + x'F_{1x'} + y'F_{1y'}) = 0.$$

A similar equation, where the factor y'^2 is replaced by x'^2, is obtained by differentiating (4) with respect to y'. Since x' and y' are not simultaneously zero, equation (3) must hold. The coefficients of ξ' and η' in equation (2) are therefore $-y'F'$ and $x'F_1'$ respectively. After adding and subtracting the expression $(y''\xi - x''\eta)F_1'$, equation (2) takes the form

$$(5) \qquad -(\omega'F_1)' + P\xi + Q\eta = 0,$$

where

$$(6) \qquad \begin{aligned} P &= y'''F_1 + (x'y'' - y'x'')F_{1x} + F_{y'xx} - F_{x'xy} + y''F_1', \\ Q &= -x'''F_1 + (x'y'' - y'x'')F_{1y} + F_{y'xy} - F_{x'yy} - x''F_1'. \end{aligned}$$

Now

$$x'P + y'Q = \frac{d}{dt}T = 0,$$

so that there exists a function F_2 such that

$$(7) \qquad P = y'F_2, \quad Q = -x'F_2.$$

Therefore, after changing the signs in equation (5) the desired equation

$$(\omega'F_1)' - \omega F_2 = 0$$

is obtained.

The function F_2 determined by equations (6) and (7) may be found explicitly from the equation

$$(x'^2 + y'^2)F_2 = y'P - x'Q.$$

On expanding the second member and collecting terms, this equation becomes

$$(x'^2 + y'^2)F_2 = (x'x''' + y'y''')F_1 + (x'x'' + y'y'')F_1'$$
$$-F'_{xx'} - F'_{yy'} + x'(F_{x'xx} + F_{x'yy}) + y'(F_{y'xx} + F_{y'yy}).$$

Now on differentiating the identity

$$x'F_{x'} + y'F_{y'} = F$$

twice with respect to x or y, the equations

$$x'F_{x'xx} + y'F_{y'xx} = F_{xx}, \quad x'F_{x'yy} + y'F_{y'yy} = F_{yy}$$

are obtained, so that F_2 is given by the equation

$$(x'^2 + y'^2)F_2 = (x'x''' + y'y''')F_1$$
$$+ (x'x'' + y'y'')F_1' + F_{xx} - F'_{xx'} + F_{yy} - F'_{yy'}.$$

In case the parameter t is the length of arc, so that $x'^2 + y'^2 \equiv 1$, the function F_2 has the simpler form

$$F_2 = F_{xx} - F'_{xx'} + F_{yy} - F'_{yy'} - (x''^2 + y''^2)F_1.$$

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ON THE DISTANCE FROM A POINT TO A SURFACE.

BY PROFESSOR E. R. HEDRICK.

(Read before the American Mathematical Society, September 5, 1907.)

THE discussion of the extrema of the distance from a point to a surface has been made the basis for the treatment of principal radii of curvature and for the classification of points on a surface by several writers.* In this connection it is interesting to notice that a variation of an example due to Peano * results in an example of a surface such that the distance from a certain point to the surface is at a minimum at a certain point of the surface for every normal plane section of the surface through that point, while the same distance is *not* at a minimum in general on the surface at the same point.

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* See, e. g., Goursat, Cours d'analyse, or English translation, no. 60 ; the statements there made are correct, the example here considered falling under the case $s^2 - rt = 0$. See also BULLETIN, vol. 13, no. 9, pp. 447, 448 ; the statements of this article differ in their spirit from those of the present article, and comparisons must be made with this understanding.

The surface

$$(1) \qquad 2x^4 - 3x^2y - x^2 - z^2 + 1 = 0$$

is *regular* † at the point (0, 0, 1), the normal at that point is the axis of z, and the square of the distance from the origin to the surface is

$$(2) \qquad D^2 = x^2 + y^2 + z^2 = 1 + (y - x^2)(y - 2x^2),$$

so that the positive determination of D in a sufficiently small circle about the origin in the (x, y) plane is real and does not vanish. An extremum of D^2 therefore corresponds precisely to the same extremum of D. But the value of D^2 given by (2) is nothing but the Peano function

$$(3) \qquad F(x, y) = (y - x^2)(y - 2x^2)$$

increased by unity. It follows that D is at a minimum at $(x = 0, y = 0)$ along any normal section of the surface, *i. e.*, for $y = kx$ and for $x = 0$; but D is not at a minimum at (0, 0). For these are the known properties of $F(x, y)$. *That the distance from a point on the normal to each normal section is at a minimum (or at a maximum) for every normal section through the foot of the normal does not show that the same distance is at a minimum (maximum) for the surface itself.*

The surface

$$(4) \qquad \begin{aligned} x^2 + z^2 - 1 + 3ye^{-1/x^2} - 2e^{-2/x^2} &= 0, \quad \text{for } x \neq 0; \\ x^2 + z^2 - 1 &= 0, \quad \text{for } x = 0; \end{aligned}$$

is continuous and regular at the point (0, 0, 1), the normal is the axis of z, and the square of the distance from the origin to the surface is

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* A. Genocchi and G. Peano, Calcolo differenziale, etc., 1884, p. xxix; German translation, Bohlmann and Schepp, 1899, p. 332.

† That is the partial derivatives of the left-hand side of (1) exist and do not all vanish at that point. It is easy to construct and to visualize the surface (1).

$$(5)\qquad \begin{aligned} D^2 &= 1 + (y - e^{-1/x^2})(y - 2e^{-1/x^2}), \quad \text{when } x \neq 0\,; \\ D^2 &= 1 + y^2, \quad \text{when } x = 0\,; \end{aligned}$$

which is precisely the function

$$(6)\qquad \begin{aligned} \phi(x, y) &= (y - e^{-1/x^2})(y - 2e^{-2/x^2}), \quad \text{when } x \neq 0\,; \\ \phi(x, y) &= y^2, \quad \text{when } x = 0\,; \end{aligned}$$

increased by unity; the function $\phi(x, y)$ given by (6) has been studied*; it is at a minimum for any analytic curve in the (x, y) plane through the point (0, 0); but it is not at a minimum in the region about (0, 0). Thus even the knowledge that the distance from a point on the normal to a surface is at a minimum at the foot of the normal for every curve cut out of the surface by an analytic cylinder through the normal, does not prove that the same distance is at a minimum for the surface itself.

COLUMBIA, MO.,
January 11, 1908.

A GEOMETRIC REPRESENTATION OF THE GALOIS FIELD.

BY DR. L. I. NEIKIRK.

(Read before the Chicago Section of the American Mathematical Society, March 30, 1907.)

THE roots of irreducible congruences were first introduced into mathematics by Galois.† Various writers since then have contributed to the theory of irreducible congruences and classes of residues.‡ The greatest progress of recent years was made by Moore.§ He proved that any finite abstract field in which division is unique is an abstract form of the Galois field; that its order is the power of a prime p^n, and that for any order it is unique, being independent of the particular irreducible congruence of degree n used in defining it.

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* *Annals of Mathematics*, vol. 8, no. 4 (July, 1907), pp. 172–174. The exact significance of "analytic," as here used, is there specified, and a more general statement of the property quoted is given.

† "Sur la théorie des nombres," *Bulletin des Sciences Math.* de M. Ferrussac (1830); also Œuvres mathématiques d'Evariste Galois, Gauthier-Villars, Paris, 1897.

‡ See the preface of Linear Groups by L. E. Dickson.

§ BULLETIN, December, 1893; Chicago Congress Mathematical Papers, pp. 208–242.

More recently Maclagan-Wedderburn* and Dickson proved simultaneously that any finite field is necessarily commutative.

The object of this paper is to put the Galois field — and, by the above, any finite field — in the concrete form of a geometric representation.†

Consider the equation $f(x) = 0$ of degree n and irreducible in the domain of rational numbers. It has a root j which we may represent in the ordinary way on the complex plane. If we plot all the points

$$(1) \qquad c_1 j^{n-1} + c_2 j^{n-2} + \cdots + c_{n-1} j + c_n,$$

the c's being integers, we get a complete geometric representation of the integers of the algebra defined by $f(x) = 0$. This set of points is closed with regard to the ordinary geometric forms of the rational operations addition, subtraction, multiplication, but not division.

The following theorem is due to Jacobi: ‡

If the vectors $\omega_1, \omega_2, \cdots, \omega_r$ $(r > 2)$ are not connected by some relation of the form

$$m_1\omega_1 + m_2\omega_2 + \cdots + m_r\omega_r = 0,$$

then integral values of the m's can be chosen making

$$|m_1\omega_1 + m_2\omega_2 + \cdots + m_r\omega_r| < \epsilon,$$

where ϵ may be any arbitrarily small positive quantity.

By this theorem all the points of the set (1) for $n > 2$ are limiting points, and the set forms an enumerable assemblage, dense around its own points.

Consider the Galois field defined by the irreducible congruence $f(x) \equiv 0 \pmod{p}$.§ We apply to this the representation given by (1) by distinguishing two kinds of points, viz., reduced points for which the c's are positive and less than p, and the remaining points called congruent points.‖ In particular, the points

$$(\bar{c}_1 + k_1 p)j^{n-1} + (\bar{c}_2 + k_2 p)j^{n-2} + \cdots + (\bar{c}_{n-1} + k_{n-1}p)j + (\bar{c}_n + k_n p)$$

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* *Transactions Amer. Math. Society*, vol. 6 (1905), pp. 349–352.

† The geometric representation of residues to a prime modulus ($n = 1$) is given by Dickson, Linear Groups, p. 3, and will not be considered here.

‡ Ges. Werke, vol. 2, pp. 27–32; also see Clebsch and Gordan's Theorie der Abel'schen Functionen, § 38.

§ This constitutes a farther restriction on $f(x)$.

‖ The exceptional case $x^2 + x + 1 \equiv 0 \pmod{2}$ noted by Galois (Works, footnote p. 17) is only an apparent and not a real exception here.

are said to be congruent to each other and to the reduced point

$$\bar{c}_1 j^{n-1} + \bar{c}_2 j^{n-2} + \cdots + \bar{c}_{n-1} j + \bar{c}_n.$$

When two reduced points are combined by the geometric forms of addition, subtraction, and multiplication the result is in general a point of set (1), not a reduced point but congruent to some definite third reduced point. This gives a geometric interpretation of the above three rational operations in the Galois field.

Let ω be a primitive root among the reduced points. Consider the points

$$\omega,\ \omega^2,\ \cdots,\ \omega^{p^n-1}\ (\equiv 1); \tag{2}$$

no two of these are congruent points. The quotient of two of these, $\omega^s \div \omega^r = \omega^{s-r}$ $(s > r)$, gives a definite point in set (2). If $r > s$, it will be necessary to multiply ω^s by ω^{p^n-1} before performing the operation of division.

The geometric operation of division in the Galois field is performed in the following manner. Points in set (2) congruent to the two reduced points given are combined by the ordinary geometric form of division; the resulting point of set (2) is congruent to a definite third reduced point. This completes the four rational operations in the Galois field.

Only a limited number of points of set (1) are used in all possible operations between the reduced points. This geometric representation of the Galois field is not unique, as an indefinitely large number of choices of j may be made. In particular, we may always chose j real if we wish, as is always possible, to make the constant term of $f(x)$ negative.

If $j = \rho(\cos\theta + i\sin\theta) = \rho\epsilon^{i\theta}$, then the elements of the Galois field are given in the form

$$\bar{c}_1\rho^{n-1}\epsilon^{i(n-1)\theta} + \bar{c}_2\rho^{n-2}\epsilon^{i(n-2)\theta} + \cdots + \bar{c}_{n-1}\rho\epsilon^{i\theta} + \bar{c}_n.$$

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CONCERNING THE DEGREE OF AN IRREDUCIBLE LINEAR HOMOGENEOUS GROUP.

BY PROFESSOR W. B. FITE.

(Read before the American Mathematical Society, September 6, 1907.)

In the *Transactions*, volume 8 (1907), pages 107–112, the writer has considered the connection between the degree of an irreducible linear homogeneous group and its abstract group properties. The discussion was limited for the most part to groups whose orders are powers of a prime. In the present paper some of the results of the former paper are extended to other groups.

THEOREM I. *A linear homogeneous group G all of whose invariant operations are similarity substitutions and such that every invariant subgroup contains invariant operations besides identity either is irreducible or is simply isomorphic with each of its irreducible components.*

If G is reducible, suppose that it has been put into its completely reduced form. The invariant operations will not be affected by this change. If any irreducible component were not simply isomorphic with G, the subgroup of G that would correspond to identity in this component would contain invariant operations besides identity. But this is impossible since every invariant operation of G is a similarity substitution. Hence every irreducible component is simply isomorphic with G.

Let G be any group of finite order g that has a cyclic central generated by the operation h of order a (>1) and when G is written as a regular permutation group, of form

$$(x_{1,1}x_{1,2}\cdots x_{1,a})(x_{2,1}x_{2,2}\cdots x_{2,a})\cdots(x_{g/a,1}x_{g/a,2}\cdots x_{g/a,a}).$$

The linear substitution S

$$y_{i,j}=\sum_{k=1}^{a}\omega^{(k-1)(j-1)}x_{i,k}\quad(i=1,2,\cdots,g/a;\ j=1,2,\cdots,a),$$

where ω is a primitive ath root of unity, transforms G into a semi-canonical form and transforms h into its normal form.*

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* *Transactions Amer. Math. Society*, loc. cit., p. 108.

In this transformed form of G the variables

$$x_{1,j},\ x_{2,j},\ \cdots,\ x_{g/a,j} \qquad (j=1,\ 2,\ \cdots,\ a)$$

are transformed into linear combinations of themselves. Let G_j be the group formed by the substitutions of G as far as they affect these g/a variables. If $j-1$ is relatively prime to a, G_j is simply isomorphic with G.

If we suppose that every invariant subgroup of G contains at least one invariant operation besides identity, it follows that G_j is simply isomorphic with each of its irreducible components. We have therefore

THEOREM II. *A necessary and sufficient condition that a group all of whose invariant subgroups contain invariant operations besides identity be simply isomorphic with an irreducible group is that its central be cyclic.*

If $j'-1$ $(j' \neq j)$ is also relatively prime to a, then since the coefficients of G_j are either zero or powers of ω^{j-1} and $G_{j'}$ differs from G_j only in having $\omega^{j'-1}$ for ω^{j-1}, it follows that for every irreducible component of G_j there is a non-equivalent irreducible component of $G_{j'}$, and conversely. Moreover if $j-1$ is not relatively prime to a, G_j is not simply isomorphic with G. Hence the number of irreducible representations of G of any degree that are simply isomorphic with it is a multiple of $\phi(a)$.

If G is simply isomorphic with an irreducible group of the maximum degree[*] $n=\sqrt{g/a}$, then every invariant subgroup of G contains invariant operations besides identity. This follows from the fact that when the regular form of G is completely reduced every irreducible component occurs a number of times equal to its degree.[†]. Moreover the number of distinct representations of G that are simply isomorphic with it is $\phi(a)$.

Consider the formula [‡]

$$\sum_R \chi(S^{-1}R^{-1}SR) = \frac{g}{f}\chi(S)\chi(S^{-1}).$$

Now suppose that G is a group in which every non-invariant commutator gives an invariant commutator besides identity,[§]

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[*] *Transactions*, vol. 7 (1906), p. 67.

[†] Burnside, *Acta Mathematica*, vol. 28 (1904), p. 383.

[‡] Frobenius, *Berliner Sitzungberichte*, 1896, II., p. 1362. We here indicate the order of the group by g instead of h.

[§] This category of groups is contained in the one defined by the property that every invariant subgroup contains invariant operations besides identity.

and let S be a non-invariant operation of G that gives no invariant commutator besides identity. It follows from the formula just given that $\chi(S) \neq 0$. That is, in an irreducible group of the kind just described a necessary and sufficient condition that the trace (Spur) of any operation be different from zero is that the operation give no invariant commutator besides identity.*

Since the trace of every non-invariant commutator is zero and S gives no invariant commutators besides identity, we have $sf^2/g = \chi(S)\chi(S^{-1})$, where s indicates the number of operations with which S is commutative. Now

$$\sum_S \chi(S)\chi(S^{-1}) = g.\dagger$$

This gives $f^2 = g^2/(hg + \Sigma s_i)$, where h indicates the number of invariant operations of G, and s_i indicates the number of operations commutative with S_i (S_1, S_2, $\cdots$ being the non-invariant operations that give no invariant commutators besides identity). But each of the g/s_i operations of the conjugate set to which S_i belongs gives the same s_i. Therefore $f^2 = g/(h + j)$, where j is the number of sets of conjugates among the non-invariant operations of G that give no invariant commutator besides identity. This formula gives, for example, the degree of any irreducible group of the third class.

If G is of order p^m (p a prime), then $\chi(S)\chi(S^{-1}) = p^{s+2n-m}$, where S is a non-invariant operation that gives no invariant commutators besides identity, p^s is the number of operations commutative with S, and p^n is the degree of G. Hence, *in an irreducible group of order p^m such that every non-invariant commutator gives an invariant commutator besides identity, the square of the absolute value of the trace of any operation is a power of p (positive or negative)*.‡

If G is any irreducible group of order p^m we have §

$$\sum_i x_i p^{2n_i} = p^{m-1}(p-1).$$

But $\sum_i x_i$ is the excess of the number of conjugate sets of G over the number of conjugate sets of $G/\overline{H}$. Hence

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* Cf. *Transactions*, vol. 8 (1907), p. 110.

† Frobenius, loc. cit., pp. 1362, 1372.

‡ Cf. *Transactions*, loc. cit., p. 110.

§ Loc. cit., p. 109.

$$\sum_i x_i = \frac{(p^a + j)(p - 1)}{p}.$$

If now G can be simply isomorphic with irreducible groups of only one degree p^n, then

$$\frac{(p^a + j)(p - 1)}{p} p^{2n} = p^{m-1}(p - 1), \qquad p^{2n} = \frac{p^m}{p^a + j}.$$

Suppose now that G is simply isomorphic with irreducible groups of just two different degrees, p^{n_1} and p^{n_2} $(n_1 < n_2)$. Then

$$x_1 p^{2n_1} + x_2 p^{2n_2} = p^{m-1}(p - 1),$$

$$x_1 + x_2 = \frac{(p^a + j)(p - 1)}{p}.$$

Since x_1 and x_2 are integers, $p^a + j$ must be a power of p, say p^r. Obviously p^{2n_1} and p^{2n_2} cannot both be less than, or both greater than p^{m-r}. Therefore let

$$p^{2n_1} = p^{m-r-t_1} \quad \text{and} \quad p^{2n_2} = p^{m-r+t_2} \quad (t_1, t_2 > 0).$$

Then

$$x_2 = \frac{p^{r-1}(p - 1)(p^{t_1} - 1)}{p^{t_1+t_2} - 1}.$$

But this cannot be an integer. Hence, *a group of order a power of a prime cannot be simply isomorphic with irreducible groups of just two different degrees.*

It follows from this that if a group of order p^m is simply isomorphic with irreducible groups of different degrees, m must be greater than 9.*

CORNELL UNIVERSITY,
January, 1908.

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*See *Transactions*, loc. cit., p. 110.

ON THE LORENTZIAN TRANSFORMATION AND THE RADIATION FROM A MOVING ELECTRON.

BY DR. F. R. SHARPE.

(Read before the American Mathematical Society, February 29, 1908.)

1. *Introduction.* The electromagnetic field produced by a moving charge has been discussed by many writers.* Langevin † in particular has given an elegant investigation of the field due to a moving electron. Poincaré ‡ has partially solved Langevin's problem by the Lorentzian § transformation. In the present paper the Lorentzian transformation is applied to the expressions for the scalar and vector potentials in the form of definite integrals instead of to the differential equations for the electromagnetic field. The field due to an electron whose velocity is zero at the instant considered is very simply determined. By means of the Lorentzian transformation, the field for the more general case when the velocity is not zero is deduced. The results are Langevin's expression for the electric force and a new expression for the magnetic force.

2. *The electromagnetic equations and their solution.* Let E denote the electric force, H the magnetic force, ρ the volume density, and v the velocity of the electrons, and let the velocity of light be the unit of velocity. Lorentz's ‖ fundamental equations are

$$(1)\qquad \operatorname{div} E = \rho, \qquad (2)\qquad \operatorname{div} \rho v = -\frac{\partial \rho}{\partial t},$$

$$(3)\qquad \operatorname{curl} H = \frac{\partial E}{\partial t} + \rho v, \qquad (4)\qquad \operatorname{curl} E = -\frac{\partial H}{\partial t}.$$

The solution of these equations is known to be reducible to the determination of a scalar potential ϕ and a vector potential a which satisfy the equations

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* See Lorentz, Encyklopädie, d. math. Wissensch., vol. V. 2, p. 174.

† *Journal de Physique*, 1905.

‡ *Circolo mat. di Palermo*, 1906.

§ Lorentz, *Amsterdam Proc.*, 1903. Einstein, *Annalen der Physik*, 1905.

‖ Loc. cit.

$$(5)\quad \nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} = -\rho, \qquad (6)\quad \nabla^2\alpha - \frac{\partial^2\alpha}{\partial t^2} = -\rho v,$$

where

$$(7)\quad E = -\frac{\partial\alpha}{\partial t} - \text{grad}\,\phi \qquad \text{and} \qquad (8)\quad H = \text{curl}\,\alpha.$$

The solutions of (5) and (6) are known to be *

$$(9)\qquad \phi = \frac{1}{4\pi}\int\int\int\frac{\rho_1 dx_1 dy_1 dz_1}{r},$$

$$(10)\qquad \alpha = \frac{1}{4\pi}\int\int\int\frac{\rho_1 v_1 dx_1 dy_1 dz_1}{r},$$

where ρ_1 and v_1 are the values of ρ and v at (x_1, y_1, z_1) at time $t-r$, r being the time from (x, y, z) to (x_1, y_1, z_1). These expressions for the potentials may be expressed in terms of the actual charges of the electrons. If we imagine a sphere whose center is (x, y, z) and whose radius r diminishes with the velocity of light and is zero at time t, an electron will affect (x, y, z) only while the surface of the sphere is passing through the electron. Hence if the radial velocity of an electron is v_r, the effect of the motion of the electron is to increase the apparent charge in the ratio 1 to $1 - v_r$. Therefore if de denotes the actual charge of an electron at (x_1, y_1, z_1),

$$(11)\quad \phi = \int\frac{de}{r(1-v_r)}, \qquad (12)\quad \alpha = \int\frac{v\,de}{r(1-v_r)}.$$

3. *The Lorentzian Transformation.* Consider the transformation

$$(13)\quad x' = \beta(x - Vt), \quad t' = \beta(t - Vx), \qquad (\beta = 1/\sqrt{1-V^2})$$

and its inverse

$$(14)\qquad x = \beta(x' + Vt'), \quad t = \beta(t' + Vx').$$

If we wish to transform the electrical system into a new electrical system with the same charges, we must have

$$(15)\qquad \int\int\int\rho' dx' dy dz = \int\int\int\rho\,dx dy dz.$$

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* Rayleigh, Theory of sound, vol. 2, p. 104.

On the left hand side of this equation t' is constant, while on the other side t is constant. On account of the velocity v_x of the electrons $(x+dx+v_x dt, t+dt)$ is tranformed into $(x'+dx', t')$. Hence from (13)

$$dx' = \beta(dx + v_x dt - V dt), \quad 0 = \beta\{dt - V(dx + v_x dt)\}.$$

Solving, we find

$$dx' = \frac{dx}{\beta(1 - Vv_x)}$$

and therefore to satisfy (15) we must have

$$(16) \qquad \rho' = \beta\rho(1 - Vv_x).$$

It is important to notice that (13) and (16) completely determine the new electrical system.

We proceed to find the values of v', ϕ', α', E', H'. Differentiating (13) we have

$$(17) \qquad v'_x = \frac{dx'}{dt'} = \frac{v_x - V}{1 - Vv_x}, \qquad v'_y = \frac{dy}{dt'} = \frac{v_y}{\beta(1 - Vv_x)}, \qquad v'_z = \frac{dz}{dt'} = \frac{v_z}{\beta(1 - Vv_x)}.$$

It is easily verified that the values of ρ' and v' just obtained satisfy the continuity equation

$$(2') \qquad \operatorname{div} \rho' v' = -\frac{\partial \rho'}{\partial t'}.$$

It should also be noticed that in (11) and (12)

$$(18) \qquad t - t_1 = r$$

or

$$(t - t_1)^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2.$$

This equation is transformed by (14) into

$$(t' - t'_1)^2 = (x' - x'_1)^2 + (y - y_1)^2 + (z - z_1)^2$$

or

$$(18') \qquad t' - t'_1 = r'.$$

Hence the new electron is in position at time t'_1 to act on the new point (x', y', z') at time t'. Further it is readily shown from (17) and (18) that

$$(19)\qquad r'(1-v'_r)=\frac{r(1-v_r)}{\beta(1-Vv_x)}.$$

Hence

$$\phi'=\int\frac{de}{r'(1-v'_r)}=\int\frac{\beta(1-Vv_x)de}{r(1-v_r)}$$

or

$$(20)\qquad \phi'=\beta(\phi-V\alpha_x),$$

and similarly

$$\alpha'_x=\beta(\alpha_x-V\phi),\quad \alpha'_y=\alpha_y,\quad \alpha'_z=\alpha_z.$$

Again from (14)

$$\frac{\partial}{\partial x'}=\beta\left(\frac{\partial}{\partial x}+V\frac{\partial}{\partial t}\right),\quad \frac{\partial}{\partial t'}=\beta\left(\frac{\partial}{\partial t}+V\frac{\partial}{\partial x}\right).$$

Therefore

$$E'_x=-\frac{\partial\alpha'_x}{\partial t}-\frac{\partial\phi'}{\partial x'}=-\frac{\partial\alpha_x}{\partial t}-\frac{\partial\phi}{\partial x}$$

or

$$(21)\qquad E'_x=E_x,$$

and similarly

$$E_{y'}=\beta(E_y-VH_z),\quad E_{z'}=\beta(E_z+VH_y),\quad H_{x'}=H_x,$$
$$H_{y'}=\beta(H_y+VE_z),\quad H_{z'}=\beta(H_z-VE_y).$$

The new electrical system is therefore completely expressed in terms of the old system.

4. *Radiation from an electron.* If the charge of the electron is e the potentials are, from (11) and (12),

$$(22)\qquad \phi=\frac{e}{4\pi r(1-v_r)},\quad \alpha=\frac{ev}{4\pi r(1-v_r)}.$$

Langevin has calculated the derivatives of ϕ and α, and so determined the electric and magnetic forces. The problem is simplified by taking the x axis in the direction of the velocity and applying the Lorentzian transformation so as to reduce the velocity to zero; solving the new system and reversing the transformation so as to obtain the solution of the original system. In this case

$$(23) \qquad V = v_x, \quad v_y = v_z = v'_x = v'_y = v'_z = 0,$$

and from (22)

$$(24) \qquad \phi' = \frac{e}{4\pi r'}, \quad \alpha' = 0.$$

The derivatives of ϕ' and α' depend on the acceleration j. On differentiating (17) and using (23), we obtain

$$(25) \qquad j'_x = \frac{dv'_x}{dt'} = \beta^3 j_x, \quad j'_y = \beta^2 j_y, \quad j'_z = \beta^2 j_z.$$

From (22)

$$\phi' + d\phi' = \frac{e}{4\pi(r' + dr')(1 - j'_r dt'_1)}, \quad d\alpha' = \frac{e j' dt'_1}{4\pi r'}.$$

But if t' is constant, (18) gives $dt'_1 = -dr'$.
Hence

$$\text{grad } \phi' = -\frac{e}{4\pi r'^3}\bar{r}' - \frac{e j'_r}{4\pi r'^2}\bar{r}'.$$

Also if r' is constant $dt'_1 = dt'$. Hence

$$\frac{\partial \alpha'}{\partial t'} = \frac{e\bar{j}'}{4\pi r'}.$$

Hence

$$E' = \frac{e}{4\pi r'^3}\bar{r}' + \frac{e j'_r}{4\pi r'^2}\bar{r}' - \frac{e\bar{j}'}{4\pi r'}.$$

Again

$$H' = \text{curl } \alpha' = \frac{\partial \alpha'_t}{\partial r'} = \frac{e}{4\pi r'^2}[\bar{j}' r'],$$

where $d\alpha'_t$ is the transverse component of $d\alpha'$. The first term of E' is the force E'_1 due to the electron at rest. The other two terms are due to the acceleration and have a resultant $e\bar{j}'_t/4\pi r' = E'_2$. This is obviously equal and perpendicular to H'. Hence E'_2 and H' are equal and r', E'_2, H' are mutually perpendicular, E'_2 being in the plane of r' and j'. If we now apply the inverse Lorentzian transformation the first term of E' gives the well-known field for a uniformly moving charge, the velocity wave of Langevin. The terms depending on the acceleration (when expressed in terms of the original coordinates the origin being taken at the electron so that $x_1 = y_1 = z_1 = 0$) are

$$E'_x = \frac{e}{4\pi(r - Vx)^3}\{(y\dot{j}_y + z\dot{j}_z)(x - Vr) - (y^2 + z^2)\dot{j}_x\},$$

$$E'_y = \frac{e}{4\pi\beta(r - Vx)^3}[\{z\dot{j}_z + \beta^2(x - Vr)\dot{j}_x\}y - \{z^2 + \beta^2(x - Vr)^2\}\dot{j}_y],$$

$$E'_z = \frac{e}{4\pi\beta(r - Vx)^3}[\{\beta^2(x - Vr)\dot{j}_x + y\dot{j}_y\}z - \{\beta^2(x - Vr)^2 + y^2\}\dot{j}_z],$$

$$H'_x = \frac{e}{4\pi(r - Vx)^2}(z\dot{j}_y - y\dot{j}_z), \qquad (26)$$

$$H'_y = \frac{e\beta}{4\pi(r - Vx)^2}\{(x - Vr)\dot{j}_z - z\dot{j}_x\},$$

$$H'_z = \frac{e\beta}{4\pi(r - Vx)^2}\{y\dot{j}_x - (x - Vr)\dot{j}_y\}.$$

Making the transformation, we easily find

$$E_x = \frac{e}{4\pi(r - Vx)^3}\{(x\dot{j}_x + y\dot{j}_y + z\dot{j}_z)(x - Vr) - r(r - Vx)\dot{j}_x\},$$

$$E_y = \frac{e}{4\pi(r - Vx)^3}[(x\dot{j}_x + y\dot{j}_y + z\dot{j}_z)y - r(r - Vx)\dot{j}_y],$$

$$(27) \quad E_z = \frac{e}{4\pi(r - Vx)^3}\{(x\dot{j}_x + y\dot{j}_y + z\dot{j}_z)z - r(r - Vx)\dot{j}_z\},$$

$$H_x = \frac{e}{4\pi(r - Vx)^3}(z\dot{j}_y - y\dot{j}_z)(r - Vx),$$

$$H_y = \frac{e}{4\pi(r - Vx)^3}\{(x\dot{j}_z - z\dot{j}_x)(r - Vx) - (x\dot{j}_x + y\dot{j}_y + z\dot{j}_z)Vz\}.$$

$$H_z = \frac{e}{4\pi(r - Vx)^3}\{(y\dot{j}_x - x\dot{j}_y)(r - Vx) + (x\dot{j}_x + y\dot{j}_y + z\dot{j}_z)Vy\}.$$

Expressed in vector notation, we have therefore

$$E = \frac{e}{4\pi\{r - (Vr)\}^3}(r\dot{j})\overline{R} - \frac{er}{4\pi\{r - (Vr)\}^2}\overline{\dot{j}},$$

which is Langevin's result, R being the distance from (x, y, z) to the point which the electron would reach at time t if its acceleration were zero; that is, a force directed towards this point and a force in the direction of the acceleration. Moreover

$$H = \frac{e}{4\pi\{r-(Vr)\}^2}[rj] - \frac{e}{4\pi\{r-(Vr)\}^3}(rj)[Vr],$$

that is, a force perpendicular to $\bar{r}$ and $\bar{j}$ and a force perpendicular to $\bar{r}$ and $\overline{V}$. It is easily verified from (27) that E is equal to H and that r, E, H are mutually perpendicular.

Cornell University,
January, 1908.

SHORTER NOTICES.

On the Resolution of Higher Singularities of Algebraic Curves into Ordinary Nodes. By B. M. Walker. Doctor Dissertation, University of Chicago, 1906. 8vo. 52 pp. + Vita.

As stated in the introduction, this dissertation completes in detail a procedure proposed by Clebsch for resolving the higher singularities of algebraic plane curves. The method is this: To relate the points of the plane, one to one, to points of a general cubic surface in such a way that one point of multiplicity higher than 2, with tangents all distinct, is distributed into ordinary points of a curve on the cubic. Then by projecting back upon the plane from a center on the cubic surface, no new singularities are introduced except ordinary double points.

The first half of the work (26 pages) is devoted to the relation of plane and cubic surface by means of a three-parameter linear system of plane cubics with six common base points. The restriction that these six points shall not lie on a conic suffices to insure that there shall be in the plane no fundamental curves, *i. e.* no curves all of whose points correspond to a single point of the cubic. On the cubic surface, however, the six base points of the plane are represented by six straight lines. The author shows in detail what plane curves give rise to the rest of the twenty-seven lines on the cubic. Six of these are conics which contain five of the base points apiece; the others are the fifteen join lines of the six base points. This study is essential in preparation for the topics which follow.

In the latter half appears the main object of this research. In the singular point to be resolved is placed one of the six base points of the plane cubics. Thus on the surface this r-fold point appears as r ordinary points on the corresponding fundamental line. Conscientiously it is proved that other multiple points do not change their essential character by the transformation. For this purpose Jordan's definition of cycle of a plane curve is happily generalized. Then are specified the five loci on the cubic surface which must be avoided in choosing a center for the projection back on the plane, and it is proved that these restrictions are sufficient for the exclusion of new higher singularities.

It should be noted that Dr. Walker takes his start from Noether's reduction of the curve to one at whose singular points the tangents are all distinct. This work is a thoroughly readable presentation of what is probably the most graphic reduction of the higher singularities.

H. S. WHITE.

Die Kegelschnitte des Gregorius a St. Vincentio in vergleichender Bearbeitung. Von KARL BOPP. Mit 329 Textfiguren. *Abhandlungen zur Geschichte der mathematischen Wissenschaften,* XX Heft, 2 Stück, pp. 83–314. Leipzig, B. G. Teubner, 1907.

THE glare of notoriety of Gregorius a St. Vincentio as a circle-squarer has cast his solid achievements into obscurity. At a time when only four of the seven books of the conics of Apollonius of Perga were known in the occident, Gregorius prepared a masterly work on conic sections, the Opus geometricum, which was published at Antwerp in 1647. He did not possess the genius of his great contemporaries, Desargues and Pascal. Yet in the endeavor to gain a detailed picture of the progress of geometry of the seventeenth century, one cannot overlook Gregorius a St. Vincentio. Through the researches of Bosmans there has been an enrichment of biographical detail of this man. To make a knowledge of his very diffuse treatise more readily accessible and to point out his relation to ancient writers, as well as his influence on the progress of geometry, Mr. Bopp has prepared the booklet under review. The chief novelty in Gregorius's geometry is the method of transformation of one conic into another, called *per subtensas* (by chords), which, in the opinion of Bopp, "even now would be serviceable for didactic purposes" (page 307), which "holds *in nuce* the entire analytic geometry of conics" (page 295) and "is fit to place him, Gregorius, among the founders of analytic geometry" (page 309). While, in our opinion, Bopp claims too much for this method, it is doubtless of historic interest. To gain an idea of it, take the following example: In subjecting the parabola to the transformation "by chords," Gregorius in one place draws a chord from the vertex to any point of the parabola and takes the length of this chord as the ordinate of a point having the same abscissa as the chosen point on the parabola. This new point lies on the equilateral hyperbola whose transverse axis lies in the geometric axis and is equal to the latus rectum of the parabola. By this process the hyperbola can be constructed by points. It is shown how to derive, *per subtensas*, the hyperbola from the ellipse and also how the hyperbola may be transformed into itself. Similarly for the other conics.

FLORIAN CAJORI.

Annuaire du Bureau des Longitudes pour l'An 1908. Paris, Gauthier-Villars.

THE *Annuaire* has been published continuously since 1796, and the present volume is the 112th of the series. This publication has been so often reviewed in the "Shorter Notices" that a notice of the special features for the current year is sufficient. In accordance with the plan adopted in 1904, it contains the tables and explanations of physical and chemical constants, those of geographical and statistical constants being inserted only in the odd-numbered years.

The special articles at the end of the volume are five in number. The first and longest is a popular account, by M. G. Bigourdan, of the methods used for obtaining the parallaxes and distances of the heavenly bodies. The author adopts a chronological treatment and gives a fairly complete historical summary without at any time wearying the reader. M. Deslandres gives an account of the meetings at St. Louis, Oxford, and Meudon of the international union for solar research, from which it is easy to see how the systematic study of a branch of physics may elevate the subject into a science. The observatory of Montsouris in France exists for educational purposes only: M. Guyon explains its methods and equipment. Many geographers and travellers have learnt there the practical astronomy necessary to those who explore far from the beaten tracks. One of its features is the absence of any regular classes; those who go there to learn can have a lesson at any time and the lesson is made to suit the special needs of the pupil. The last two articles are obituary notices of M. Loewy by H. Poincaré; and of C. Trépied by M. Loewy, written shortly before the sudden death of the latter.

ERNEST W. BROWN.

Die Zustandsgleichung der Gase und Flüssigkeiten und die Kontinuitätstheorie. Von J. P. KUENEN. Braunschweig, Vieweg und Sohn, 1907. x + 241 pp.

KUENEN's volume on the equation of state of gases and fluids is one of the twenty monographs already printed by Vieweg and Son under the general title "Die Wissenschaft." It must be obvious to all that the subject of this particular volume is one which lends itself well to treatment in a separate monograph. The main outlines of the theory of corresponding states are given in many books, and no book which has to do with gases or fluids can get on without the equation of state; but the details of those theories and their agreement or disagreement with the results of the hundreds of experiments hich have been performed since they were broached are not to found collected in general texts and require for their satis- tory treatment a monograph like this.

The first five chapters, of about ten pages each, may be said e of a heuristic and qualitative nature. The author traces history of the rise of observations on the phenomena of nsation and on the existence of some principle of con- between the different states of matter. He touches he kinetic theory sufficiently to show the justification of law from that point of view and to indicate how van als was led to his equation. After obtaining that equa- author goes on to a careful explanation of phenomena nsation and of the principle of continuity from the basis furnished by the equation. At about this point there begin to appear numerous evidences of the great care with which matters are to be set forth in their true light, and of the conscientious criticism with which the author is to expound the relation between experiment and theory. For instance, it is pointed out that although van der Waals's equation may be deduced from molecular considerations, yet in a broad way the theory of continuity does not depend on such considerations; it is noted that the vapor pressure is not strictly constant but has been found to depend on the relative amounts of liquid and vapor which are present; and, moreover, the effects of gravity and slight impurities are discussed and found to be surprisingly large for states of the mixture near the critical point.

The next four chapters fill about eighty pages and may be considered as the main part of the text. They deal with the correlation of the equation of state with experiment. Citations from experimental researches and deductions from thermodynamic theory as applied to van der Waals's equation are abundant and are freely intermingled until the reader cannot fail to see their interrelations, their points of concord, and their remaining outstanding differences. The critical constants are derived by two methods and put into relation with the theory of corresponding states. The wide divergence of the theoretical value of $RT/pv = 2.67$ at the critical point and actual values found by Young is amply illustrated by giving Young's results in detail. The influence of the temperature on the "constants" of van der Waals's equation is discussed, and some very neat methods (with figures) are given for readily comparing the theoretical and experimental isothermal curves. For handling the phenomena of saturation several distinct methods, including those of free energy and the thermodynamic potential, are offered and somewhat extensive tables afford a means of checking the results against actual observations. The chapter on thermal quantities (thermische Grössen) is equally detailed and instructive; there is really nothing more to be said about it.

The tenth chapter is a note of two pages on the dimensions of molecules. The following chapter deals with corresponding states. The distinction between substances which are normal and abnormal is brought out, and a number of somewhat unusual topics such as the determination of the degree of association, viscosity, capillarity, and molecular refraction are included in the discussion. The remaining three chapters are concerned with the improvements which have been suggested for van der Waals's equation, and the allied considerations. Among others, the equations of Lorentz, Reinganum, and Clausius are developed. The last of these chapters bears the title "Mathematical methods of deriving the equation of state" and is naturally the most mathematical and difficult of the whole book. It should be stated, however, that the author nowhere shuns the use of that small amount of mathematics which is really essential to the proper development of his subject.

From this brief summary, it may be gathered that the range of topics which is treated is not large but that the treatment given is thorough, highly competent, open minded, and impartial. In short the book is just what it claims to be—a monograph in which the most important if not all of our knowledge, whether theoretical or experimental, on the equation of state is collected and carefully edited. For those who are interested in this subject the book appears to be well nigh indispensable, and for those who are not yet interested it would offer a pleasant day's reading in one of the most entertaining fields of modern physics. It would not do to close this review without mentioning the extensive bibliographical lists which follow many of the chapters. These will save the student from many unhappy hours spent in trying to find the important literature of the subject.

E. B. WILSON.

NOTES.

THE Fifteenth Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Illinois, on Thursday and Friday, September 10–11, 1908.

THE following additional associate editors of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY have been appointed: Professor G. A. BLISS, of Princeton University; Professor F. R. MOULTON, of the University of Chicago; Professor E. J. WILCZYNSKI, of the University of Illinois.

AT the meeting of the London mathematical society held on February 13, the following papers were read: By H. A. DE S. PITTARD, "Proof that every algebraic equation has a root"; by W. H. YOUNG, "On the uniform approach of a continuous function to its limit"; by F. H. JACKSON, "Note on q-differences"; by A. E. WESTEN, "An extension of Eisenstein's law of reciprocity (second paper)"; by E. CUNNINGHAM, "Conformal representation and the transformation of Laplace's equation."

A REGULAR meeting of the Syracuse section of the association of teachers of mathematics in the middle states and Maryland was held at the Syracuse high school, February 22. After a word of greeting by Principal HARPER, the following papers were read: By W. G. RAPPLEYE, "The teaching of elementary mathematics from a rational standpoint"; by G. R. STALEY, "The purpose of problems in plane geometry." The following officers were elected for the current year: President, W. G. RAPPLEYE; secretary and treasurer, D. PRATT. The time and place of the next meeting have not yet been definitely fixed. Sixty-nine names were added to the list of members.

THE subject of Professor POINCARÉ's conference at the Rome congress has been announced as "Les mathématiques de l'avenir." On account of his health, Professor KLEIN has decided not to attend the congress. His address "Ueber die mathematische Encyklopädie," will be delivered by Professor V. DYCK.

THE following Americans expect to attend the Rome Congress: E. W. Davis, T. S. Fiske, A. B. Frizell, J. G. Hardy, C. L. E. Moore, E. H. Moore, C. A. Noble, G. D. Olds, D. E. Smith, J. M. Van Vleck, and W. D. A. Westfall.

THE committee on methods of instruction which was proposed at the Dresden meeting of the Mathematiker-Vereinigung to investigate and rearrange the courses of study in the German secondary schools has now been organized and a definite plan of procedure adopted. The committee, which consists of one or two representatives of nearly every scientific society of Germany, will consider the whole field of instruction in mathematics and the natural sciences.

THE next meeting of the British association for the advancement of science will be held at Dublin, September 2–9, under the presidency of Mr. FRANCIS DARWIN. Dr. W. N. SHAW is chairman of section A, mathematics and physics.

THE academy of sciences of Vienna has advanced M. 1000 to assist in the publication of the writings of the late Professor L. BOLTZMANN.

IN order to obtain as complete an edition of the mathematical works of PAOLO RUFFINI as possible, a general request is expressed that any letters, documents, and notices of every kind, that relate to his life, be forwarded to Professor B. G. Guccia in Palermo.

ON the occasion of the dedication of the new buildings of the physical society of Frankfurt it was announced that the heirs of the late EUGEN TORNOW had presented to the academy (Akademie für Sozial- und Handelswissenschaften) the sum of 470,000 marks, to be used for advancing instruction in mathematics and the natural sciences.

THE publishing house of B. G. Teubner in Leipzig announces that the second edition of the German translation of Pascal's Repertorium der höheren Mathematik is in the press. The first volume, which will soon appear, has been fundamentally revised with the assistance of the author and various other mathematicians under the general editorship of Dr. P. EPSTEIN, of the University of Strassburg.

DURING the year 1907 the following candidates received the degree of doctor of science with mathematics as the major subject at the University of Paris: M. FATON, "Séries trigonométriques et série de Taylor"; M. TRAYNARD, "Sur les fonctions théta de deux variables et les surfaces hyperelliptiques"; M. MONTEL, "Sur les suites infinies de fonctions"; M. LAMBERT, "Sur les coefficients du développement de la fonction perturbatrice."

THE following advanced courses in mathematics are announced for the academic year 1908–1909:

COLUMBIA UNIVERSITY. — By Professor T. S. FISKE: Advanced calculus; introduction to the theory of functions of a real variable, three hours; Functions defined by linear differential equations, three hours. — By Professor F. N. COLE: Introduction to the theory of functions, three hours; Theory of plane curves, three hours. — By Professor James MACLAY: Elliptic functions, three hours; Application of the calculus to the theory of surfaces and curves in space, three hours. — By Professor D. E. SMITH: History of mathematics, two hours. — By Professor C. J. KEYSER: The principles of mathematics, three hours; Modern theories in geometry, three hours. — By Professor H. B. MITCHELL: Differential equations, two hours; Geometrical analysis, three hours. — By Professor Edward KASNER: Geometry of dynamical systems, two hours. — By Dr. G. H. LING: Theory of numbers, three hours, first half year; Advanced theory of numbers, three hours, second half year.

HARVARD UNIVERSITY. — By Professor W. E. BYERLY: Introduction to modern geometry and modern algebra, three hours; Trigonometric series (with Professor Peirce), three hours. — By Professor B. O. PEIRCE: Methods in mathematical physics, Elasticity, two hours. — By Professor W. F. OSGOOD: Differential and integral calculus (second course), three hours; Infinite series and products (first half year), three hours; Galois's theory of equations (second half year), three hours; Theory of functions (advanced course). — By Professor M. BÔCHER: Theory of functions (introductory course) three hours; The linear differential equations of physics, three hours. — By Professor C. L. BOUTON: Hydromechanics (second half year), three hours; Differential equations, Lie's theory of continuous groups, three hours. — By Professor J. K. WHITTEMORE: Elements of mechanics, three hours; Differential geometry of curves and surfaces (first half year), three hours. — By Professor E. V. HUNTINGTON: The fundamental concepts of mathematics, three hours. — By Dr. J. L. COOLIDGE: Line geometry (first half year) three hours. — By Dr. H. N. DAVIS: Dynamics of a rigid body, three hours.

Courses of reading and research are offered by Professors BYERLY, OSGOOD, BÔCHER, BOUTON, and WHITTEMORE; and a seminary in geometry will be conducted by Professor BOUTON, Professor WHITTEMORE, and Dr. COOLIDGE during the second half year.

PRINCETON UNIVERSITY. — All courses are three hours a week. The Roman numerals refer to the first (I) and second (II) term. — By Professor H. B. FINE: Theory of algebraic functions, I. — By Professor H. D. THOMPSON: Historical readings in infinitesimal geometry, I. — By Professor J. H. JEANS: Electricity and magnetism, I, II; Molecular dynamics and the kinetic theory, I, II. — By Professor G. A. BLISS: Linear differential equations, I; Partial differential equations, II. — By Professor L. P. EISENHART: Differential geometry, I, II. — By Professor W. GILLESPIE: Theory of substitutions, I; Theory of invariants, II. — By Professor O. VEBLEN: Projective geometry, I, II. — By Professor J. W. YOUNG: Theory of functions of a complex variable, I, II; Theory of numbers, I. — By Professor BLISS or VEBLEN: Theory of functions of a real variable, I, II. — By Dr. J. G. HUN: Analytic projective geometry, I, II. — By Dr. C. R. MACINNES: Elliptic functions, II. — By Dr. R. L. MOORE: Foundations of geometry, II. — By Dr. C. E. STROMQUIST: Calculus of variations, II. — By Dr. E. SWIFT: Theory of capillarity, II.

THE following courses are announced for the summer semester of 1908:

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN: Integral calculus, five hours; Conformal representation and linear differential equations, four hours; Foundations of geometry, two hours; Seminar, one and one half hours. — By Professor H. v. SEELINGER: Theory of probabilities and method of least squares, two hours. — By Professor A. VOSS: Analytic geometry of space, four hours; Analytic mechanics, II, four hours; Seminar, two hours. — By Professor A. PRINGSHEIM: Selected chapters in the theory of analytic functions, four hours; Elementary theory of ordinary differential equations, three hours. — By Professor K. DOEHLEMANN: Descriptive geometry, II, with exercises, five hours; Synthetic geometry, II, four hours; Imaginary elements of geometry, one hour. — By Professor H. BRUNN: Elements of higher mathematics, four hours. — By Dr. F. HARTOGS: Theory of finite groups and algebraic equations, four hours. — By Dr. O. PERRON: Solid geometry and spherical trigonometry, two hours; Calculus of variations, two hours.

UNIVERSITY OF STRASSBURG. — By Professor Th. REYE: Geometry of position, four hours; Seminar, two hours. — By Professor H. WEBER: Definite integrals and introduction to the theory of functions, four hours; Encyclopedia of elementary mathematics, two hours; Seminar, two hours. — By Professor J. WELLSTEIN: Introduction to the theory of invariants, four hours; Ultraelliptic functions, two hours; Seminar, two hours. — By Professor H. E. TIMERDING: Analytic geometry of space, four hours; with exercises, one hour; Mechanics, four hours. — By Dr. P. EPSTEIN: Introduction to the theory of numbers, two hours. — By Professor M. SIMON: History of mathematics in ancient times, two hours.

MR. W. W. WALLACE has been appointed professor of mechanics at the University of Liverpool.

MR. D. K. PICKEN, of the University of Glasgow, has been appointed professor of mathematics at Victoria College, Wellington, New Zealand.

DR. R. FUETER, of the University of Marburg, has been appointed associate professor of mathematics at the University of Basel.

DR. E. MEYER has been appointed docent in projective and descriptive geometry at the technical school of Charlottenburg.

DR. F. DALWIGK, of the University of Marburg, has been promoted to an associate professorship of mathematics.

AT Columbia University Professor H. B. MITCHELL has been advanced to a full professorship of mathematics.

AT the University of Chicago, Professors J. W. A. YOUNG and H. E. SLAUGHT have been promoted to associate professorships of mathematics; Professors K. LAVES and F. R. MOULTON have been promoted to associate professorships in astronomy.

MR. W. P. RUSSELL, of Pomona College, Claremont, Calif., has been promoted to an assistant professorship of mathematics.

DR. ROXANNA VIVIAN, of Wellesley College, who has been during the past two years professor of mathematics at the American College for Girls in Constantinople, has been promoted to an associate professorship of mathematics at Wellesley and will resume her work there next fall.

PROFESSOR HEINRICH MASCHKE, of the University of Chicago, died March 1, 1908, at the age of 55 years. He was born in Breslau, Germany, graduated from the University of Göttingen, and had been associated with the University of Chicago since 1892, having been made full professor of mathematics in 1907. He contributed extensively to the theory of linear groups and differential invariants. He had been a member of the AMERICAN MATHEMATICAL SOCIETY since 1893, and was a vice-president of the Society in 1907.

The following catalogues of second-hand mathematical books have recently appeared: K. F. Koehler's Antiquarium, Leipzig, Kurprinzstrasse 6, catalogue No. 572, 2586 titles in pure and applied mathematics; catalogue No. 573, 546 titles of astronomical, mathematical and physical periodicals. — Buchhandlung Gustav Fock, Leipzig, Schlossgasse 7, catalogue No. 325, 2275 titles in mathematics, physics and astronomy. — W. Heffer and Sons, Cambridge, England, catalogue No. 35, 2825 titles in mathematics and physical sciences. — James Thin, Edinburgh, 54–55 South Bridge, catalogue No. 158, about 1000 titles in mathematics, astronomy, and physical science.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Bertrand (G.). Sur une fonction déduite de la fonction O de M. Appell. Les rapports avec la fonction σ. (Thèse.) Paris, 1908. 8vo. 36 pp.

Crathorne (A. R.). Das räumliche isoperimetrische Problem. (Diss.) Göttingen, 1907. 8vo. 59 pp.

Cutroni (F.). Alcuni teoremi di geometria negli spazii a n dimensioni. Nota I (Parallelismo e perpendicolarità). Messina, d'Angelo, 1908. 8vo. 10 pp.

Dantscher (V. von). Vorlesungen über die Weierstrasssche Theorie der irrationalen Zahlen. Leipzig, Teubner, 1908. 8vo. 6 + 80 pp. Cloth. M. 3.40

Encyclopédie des sciences mathématiques pures et appliquées. Edition française publiée sous la direction de J. Molk. Tome I; deuxième volume; premier fascicule: Les fonctions rationnelles; exposé, d'après l'article allemand de E. Netto, par R. Le Vavasseur. Paris, Gauthier-Villars, 1908. 8vo. 232 pp.

Gubski (V.). Ueber den Zusammenhang zwischen den partikulären Lösungen der einzelnen Gebiete bei der hypergeometrischen Differentialgleichung dritter Ordnung mit zwei endlichen singulären Punkten. Berlin, Trenkel, 1907. 8vo. 60 pp. M. 2.00

Hansen (O.). Ueber die äquiforme Geometrie im Bündel. (Diss.) Kiel, 1907. 8vo. 49 pp.

Heiberg (J. L.) und Zeuthen (G. H.). Eine neue Schrift des Archimedes. Leipzig, Teubner, 1907. 8vo. 2 + 43 pp. M. 1.60

Hellinger (E.). Die Orthogonalinvarianten quadratischer Formen von unendlichvielen Variabeln. (Diss.) Göttingen, 1907.

Hellwig (M.). Untersuchung über die Lage der Inzidenzpunkte bei Reflexion und Refraktion an Ebene, Kugel und Kreiszylinder für zwei feste Punkte im Raum (Licht- und Augenpunkt). (Diss.) Rostock, 1906. 8vo. 30 pp.

Hüttig (F.). Arithmetische Theorie eines Galoisschen Körpers. (Diss.) Marburg, 1907. 8vo. 40 pp.

Kiepert (L.). Grundriss der Differential- und Integral-Rechnung. 2ter Teil: Integral-Rechnung. 9te verbesserte und vermehrte Auflage des gleichnamigen Leitfadens von M. Stegemann. Hannover, Helwing, 1908. 8vo. 23 + 738 pp. Cloth. M. 13.50

——. Tabelle der wichtigsten Formeln aus der Integral-Rechnung. [Aus: Grundriss der Integral-Rechnung, 9te Auflage.] Hannover, Helwing, 1908. 8vo. 59 pp. M. 0.50

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THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and thirty-seventh meeting of the Society was held in New York City on Saturday, February 29, 1908, extending through the usual morning and afternoon sessions. The attendance included the following thirty-four members:

Professor G. A. Bliss, Professor Joseph Bowden, Professor E. W. Brown, Professor F. N. Cole, Professor L. P. Eisenhart, Mr. G. W. Hartwell, Dr. A. M. Hiltebeitel, Professor W. H. Jackson, Professor Edward Kasner, Mr. E. H. Koch, Mr. W. C. Krathwohl, Dr. G. H. Ling, Mr. L. L. Locke, Professor Max Mason, Mr. A. R. Maxson, Professor H. B. Mitchell, Dr. R. L. Moore, Mr. H. W. Reddick, Professor L. W. Reid, Mr. L. P. Siceloff, Mr. F. H. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Dr. W. M. Strong, Dr. Elijah Swift, Professor H. D. Thompson, Miss M. E. Trueblood, Professor J. M. Van Vleck, Professor Oswald Veblen, Mr. H. E. Webb, Professor A. G. Webster, Professor H. S. White, Professor J. W. Young.

The President of the Society, Professor H. S. White, occupied the chair, being relieved at the afternoon session by Professor P. F. Smith. The Council announced the election of the following persons to membership in the Society: Mr. E. G. Bill, Yale University; Mr. C. H. Currier, Brown University; Mr. W. S. Pemberton, State Normal School, Edmond, Okla.; Professor S. W. Reaves, University of Oklahoma; Mr. L. L. Silverman, University of Missouri; Mr. W. M. Smith, Lafayette College; Mr. H. W. Stager, University of California. Twelve applications for membership in the Society were received.

The By-Laws of the Society were amended to provide that the appointment of members of the Editorial Committee of the *Transactions* should be made at the April meeting of the Council and that each new member should take office on the following October 1.

The usual informal dinner was arranged for the evening, and was attended by twelve members.

The following papers were read at this meeting:

(1) Professor R. D. CARMICHAEL: "On the numerical factors of certain arithmetic forms."

(2) Professor R. D. CARMICHAEL: "On the remainder term in a certain development of $f(a+x)$."

(3) Dr. F. R. SHARPE: "The Lorentzian transformation and the radiation from a moving electron."

(4) Professor VIRGIL SNYDER: Normal curves of genus 6 and their groups of birational transformations."

(5) Professor J. W. YOUNG: "The geometry of chains on a complex line."

(6) Professor J. W. YOUNG: "A fundamental invariant of discontinuous ζ-groups defined by a normal curve of order n in space of n dimensions."

(7) Professor E. B. VAN VLECK: "On non-measurable point sets."

(8) Professor L. E. DICKSON: "On higher congruences and modular invariants."

(9) Professor MAX MASON: "Note on Jacobi's equation in the calculus of variations."

(10) Dr. G. W. HILL: "Subjective geometry."

(11) Professor G. A. BLISS: "A method of deriving Euler's equation by means of an invariant integral."

(12) Professor C. N. HASKINS: "On the second law of the mean."

(13) Professor EDWARD KASNER: "The contact transformations of mechanics."

(14) Professor EDWARD KASNER: "The plane sections of an arbitrary surface."

(15) Professor G. A. MILLER: Note on the periodic decimal fractions."

(16) Dr. F. R. SHARPE: "The inner force of a moving electron."

(17) Dr. W. H. ROEVER: "Brilliant points of curves and surfaces."

(18) Mr. FRANK IRWIN: "Transformations of the elements $x, y, y', \cdots, y^{(k)}$ that carry a union of such elements over into a union."

Mr. Irwin's paper was communicated to the Society through Professor Bouton. In the absence of the authors, Professor Haskins's paper was read by Professor Mason, Dr. Sharpe's second paper was read by Professor Snyder, and Dr. Sharpe's first paper and the papers of Professor Carmichael, Professor

Van Vleck, Professor Dickson, Dr. Hill, Professor Miller, Dr. Roever, and Mr. Irwin were read by title.

Professor Van Vleck's paper will appear in the April *Transactions*. The papers of Professor Dickson, Professor Mason, and Dr. Hill appeared in the April BULLETIN. Professor Young's second paper and Dr. Sharpe's second paper appear in the present number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper Professor Carmichael obtains several propositions concerning the numerical factors of arithmetic forms determined thus: Let $Q_n(x) = 0$ be the equation whose roots are the primitive nth roots of unity. In $Q_n(x)$ put $x = \alpha/\beta$, α and β being relatively prime positive integers and $\alpha > \beta$. Multiply this expression by β raised to a power equal to the degree of $Q_n(x)$. A study is made of the factors of the resulting forms. Several propositions are proved which are generalizations of those obtained by Dickson in an article "On the cyclotomic function" (*American Mathematical Monthly*, volume 12, pages 86–89).

2. The series which Professor Carmichael studies in this paper, with the remainder term which he finds, is as follows:

$$
\begin{aligned}
f(a+x) = f(a) &+ \frac{x}{m}\left[f'\left(a+\frac{x}{m}\right)+f'\left(a+\frac{2x}{m}\right)+\cdots\right.\\
&\left.+f'\left(a+\frac{m-1}{m}x\right)+f'(a+x)\right]-\frac{x}{m\cdot 2}\,[f'(a+x)-f'(a)]\\
&-\frac{B_1x^2}{m^2\cdot 2\,!}\,[f''(a+x)-f''(a)]+\frac{B_2x^4}{m^4\cdot 4\,!}\,[f^{\mathrm{IV}}(a+x)-f^{\mathrm{IV}}(a)]\\
&-\cdots+(-1)^n\frac{B_nx^{2n}}{m^{2n}\cdot(2n)\,!}\,[f^{(2n)}(a+x)-f^{(2n)}(a)].\\
&+x^{2n+2}\left[\frac{1}{(2n+2)\,!}+\frac{1}{2m(2n+1)\,!}+\frac{B_1}{m^2\cdot 2!(2n)!}\right.\\
&\left.+\frac{B_2}{m^4\cdot 4\,!(2n-2)\,!}+\cdots+\frac{B_{n-1}}{m^{2n-2}\cdot(2n-2)\,!\;4\,!}+\frac{B_n}{m^{2n}\cdot(2n)\,!\;2\,!}\right]\\
&\times\;[f^{(2n+2)}(a+\phi_1x)-f^{(2n+2)}(a+\phi_2x)],
\end{aligned}
$$

B_1, B_2, $\cdots$ being Bernoulli's numbers, and ϕ_1 and $\phi_2 < 1$ and > 0.

This series, without remainder term, was given by S. A. Corey in the *Annals of Mathematics*, second series, volume 5, number 4 (July, 1904).

3. The field due to a moving charge has been discussed by many writers: Langevin in particular has given an elegant investigation of the field due to a moving electron. Poincaré has partially solved Langevin's problem by the use of the Lorentzian transformation. In Dr. Sharpe's paper the Lorentzian transformation is applied to the definite integral expressions for the scalar and vector potentials of the field instead of to the differential equations of the field. By aid of the transformation the problem of Langevin is reduced to the simpler problem where the velocity of the electron vanishes at the instant considered. By transforming back again, the solution of Langevin's problem is obtained with a new form for the magnetic force.

4. In Professor Snyder's paper, curves of genus 6 are defined as the partial intersection of five particular quadric spreads in space of five dimensions, R_5, and one arbitrary spread. By proper projection, this curve of order 10 can be reduced to a plane sextic with four double points. The coordinate spaces become adjoint cubics, in terms of which an equation of any sextic of this type can be written. Every possible birational transformation of this curve into itself can be generated by four linear operators of order two. The most interesting groups are G_{120}, to which a sextic with four double points at the vertices of a quadrangle belongs, a G_{150} leaving invariant a non-singular quintic (not reducible to a sextic), and a cyclic G_{26} of a certain hyperelliptic curve. The study of the non-singular quintic from this standpoint involves that of the Veronese surface in R_5.*

5. A chain in the ordinary complex projective geometry may be defined analytically as any class of points projective with the real points of a line, from which follows that there is one and only one chain through three given points of a line. With the usual representation of complex numbers on a sphere a

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* Wiman, *Bihang till k. Sven. Vet. Akad. Handlingar*, vol. 21 (1895), two articles. Clebsch, Geometrie, vol. 1, p. 695 ff. Segre, *Ann. di mat.* (2), vol. 22 (1894).

chain is simply a circle. A chain may also be defined directly in terms of three points on a line by means of the notion of a net of rationality on a line and order relations, as was done in a paper presented to the Society by Professors Veblen and Young at its last meeting. In his first paper Professor Young develops some of the fundamental properties of the chains of a line by purely synthetic methods, the results being well known in the theory of functions of a complex variable when the chains are identified with the circles on the sphere of the complex variable, or the Argand plane. The proofs are in many cases much simpler than the usual analytic proofs. Moreover it appears important to emphasize the fact that the function-theoretic theorems in question form an elementary chapter in projective geometry.

The system of chains through two given points M, N is called the system $T(M, N)$. Through any point P distinct from M, N there is one and only one chain cutting each of the chains through M, N in points harmonic with M, N. Every such chain is said to be "about" M, N; the system of all chains about M, N is denoted by $A(M, N)$. The two systems T and A are fundamental in the classification of projectivities with distinct double points. If a chain is about two points M, N, these points are said to be conjugate with respect to the chain, and one is the inverse of the other. Then if every point on a chain is defined as its own inverse, the existence and uniqueness of the inverse of any point of a line with reference to a given chain follows. Two chains are defined as orthogonal if one contains two points conjugate with respect to the other. The projectivities on a line are then classified, with reference to their invariant figures, into involutoric, elliptic, hyperbolic, loxodromic, and parabolic projectivities.

11. In the paper of Professor Bliss a method of deriving Euler's equation in the calculus of variations by means of the theory of invariant integrals of the form

$$\int\left\{A(x, y)+B(x, y)\frac{dy}{dx}\right\}dx$$

is exhibited. The method permits one to derive very simply the result of Du Bois-Reymond and Hilbert, that for a regular problem the minimizing curve can not have corner points where the slope is discontinuous. A considerable simplification in the presentation of the whole theory of the simplest problem in the plane is also effected, because the notion of an invariant integral is one which can be frequently applied in the derivation of the further conditions.

12. The second law of the mean is fundamental in the classical Dirichlet discussion of Fourier's series. The fundamental theorem of Fourier's constants, however, as proved by de la Valleé Poussin and Hurwitz, does not involve the second law of the mean. In Professor Haskins's note it is shown that the second law of the mean is a simple consequence of the fundamental theorem of Fourier's constants.

13. In the *Leipziger Berichte* for 1889 Lie studied those infinitesimal contact transformations whose characteristic function is of the form $\Omega(x, y)\sqrt{1 + p^2}$ and indicated their importance in dynamics. In the present note Professor Kasner shows that the alternant of any two transformations of this type is necessarily a point transformation. The result holds for any number of variables and may be applied to many allied theories.

14. In connection with any surface S we may consider the curves cut out by the ∞^3 planes of space. By projecting these orthogonally or centrally upon a fixed plane we obtain the triply infinite systems which are investigated in Professor Kasner's paper. Some of the properties derived are analogous to those which hold for dynamical trajectories: in particular the focal locus is circular for both types. The general classes are however distinct, and overlap only when the surface S is cylindrical or conical. The developable surfaces give rise to especially simple results. The theory worked out belongs to general projective geometry.

15. The group G formed by the $\phi(n)$ numbers which are less than n and prime to n, when they are combined with respect to multiplication modulo n, has been called by H. Weber the most important example of an abelian group of finite order. The object of Professor Miller's note is to point out how several fundamental theorems relating to the periods of decimal fractions may readily be proved by means of this group. Assuming $n > 1$ and prime to 10, it results that 10, or

its residue modulo n, is a number in G and that the powers of 10 generate a cyclic subgroup K whose order k is the length of the period of $1/n$, since K includes 1 and multiplying a decimal fraction by 10 simply moves its decimal point one place to the right. The operators of K have therefore the characteristic property that they simply affect the position of the decimal point of any number divided by n with which they are combined under G, moving it respectively $1, 2, \cdots, k$ places to the right. If these operators are applied a second time in order they will move this point respectively $k + 1$, $k + 2$, $\cdots$, $2k$ places to the right. As the k distinct numbers modulo n obtained in the second case are the same as those obtained in the first, and each of the $\phi(n)$ numbers divided by n gives rise to a different decimal fraction it results that k is the length of the period of *every* number of G divided by n.

From the fact that G always involves -1 it follows that the numbers of G may be divided into pairs differing only with respect to sign, and hence the mantissa of the one divided by n may be obtained by subtracting from g each digit in the mantissa of the other divided by n. If K involves -1, the mantissas of the two numbers differing only in sign divided by n may be obtained from each other by a cyclic permutation and hence the entire period of such a number may be obtained by subtracting from g each digit of half the period. When the index of K under G is odd, in particular when 10 is a primitive root of n, K must involve -1 and hence half the period of every number of G divided by n may be obtained by subtracting from g the digits of the other half in order. The index of K under G is the number of distinct periods in the fractions obtained by dividing by n all the numbers which are prime to n. If the mantissa of a number of G divided by n were composed of g's, that of the other number of the pair would be composed of o's; *i. e.*, the otherwise evident fact that g cannot constitute the period of a proper fraction in decimal form may be regarded as a special case of the given theorems. With the exception of the periods o and g, any arbitrary set of numbers may evidently be the period of a proper fraction.

17. In this paper, which is an extension of one reported in the BULLETIN (series 2, volume 8 (1901–02), page 279) and published in the *Annals of Mathematics* (series 2, volume 3 (1901–02), page 113), Dr. Roever begins with a description of

the optical phenomenon which suggested the problems considered. A brilliant point of a curve c with respect to two congruences is defined as a point P of c at which the curves of the two congruences which pass through P make equal angles with c at P. A system of three congruences is then considered, and the condition found that a point P should be a brilliant point of the curve of one of these which passes through P, with respect to the other two. The condition consists in the vanishing of a certain expression Ω, which involves the functions that appear in the systems of differential equations of the congruences. It is found that the expression Ω possesses a certain kind of invariance. Thus, if each of the three congruences represented in Ω be replaced by a certain congruence involving an arbitrary function, the expression Ω formed from the new congruences differs from that formed from the old ones by a nonvanishing factor. Therefore the equation $\Omega = 0$ is the locus of the brilliant points of each of an infinite number of congruences with respect to each of an infinite number of pairs of congruences. Two special kinds of brilliant point are considered in great detail. In terms of either of these, the brilliant point of a surface with respect to a pair of congruences, may be defined by considering a congruence normal to the surface. It is shown that under certain restrictions brilliant points of both curves and surfaces, may be regarded as points of contact. It is also shown that optical interpretations of one or another kind may be given to most of the results. Numerous examples illustrate the results.

18. Mr. Irwin shows that the only transformations of the elements $(x, y, y', \cdots, y^{(k)})$ carrying a union of these elements over into a union are extended contact transformations.

F. N. Cole,
Secretary.

THE FEBRUARY MEETING OF THE SAN FRANCISCO SECTION.

THE thirteenth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held on Saturday, February 29, 1908, at Stanford University. Professor Hoskins presided. The following thirteen members were present:

Professor R. E. Allardice, Professor H. F. Blichfeldt, Professor R. L. Green, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Mr. Joseph Lipke, Dr. J. H. McDonald, Professor W. A. Manning, Professor Irving Stringham, Mr. W. H. Stager, Mr. J. D. Suter, Professor S. D. Townley.

The following papers were read at this meeting:

(1) Professor D. N. LEHMER: "A discussion by synthetic methods of the covariant conic of two given conics."

(2) Mr. JOSEPH LIPKE: "Ein Fehler im König'schen Beweise des Reciprocitätsgesetzes in der Theorie der quadratischen Reste."

(3) Professor G. A. MILLER: "Generalization of the positive and negative numbers." (By title.)

(4) Professor L. M. HOSKINS: "A general diagrammatic method of representing propositions and inference in the logic of classes."

(5) Dr. J. H. McDONALD: "On certain types of continued fractions considered from a common point of view."

(6) Mr. J. D. SUTER: "On the surface $F_1(u) = u/(1 - u^4)$, $F_2(v) = v/(1 - v^4)$" (preliminary report).

(7) Professor H. F. BLICHFELDT: "On a certain basis of geometry" (preliminary report).

(8) Professor L. M. HOSKINS: "General algebraic solutions in the logic of classes."

Abstracts of the papers follow below in the order as given above.

1. Professor Lehmer gave a discussion by synthetic methods of the problem: To find the locus of a point in the plane such that the four tangents from it to two fixed conics shall be harmonic. The proof is made to depend on the theorem: Given a point row of the second order and a pencil of rays of the second order projective to it, at most four rays of the pencil

will pass through the points on the point row which correspond to them.

2. In the *Acta Mathematica*, volume 22, pages 181–191, Professor König gives a proof of the theorem of reciprocity in the theory of quadratic residues, a proof by complete induction but without the use of the famous gaussian lemma, according to which, if p is a prime of the form $8n + 1$, there always exists a prime q less than $2\sqrt{p} + 1$ such that p is a quadratic nonresidue of q. In this note, Mr. Lipke points out an error in Professor König's work which invalidates the latter's proof of the theorem, and also illustrates by an example that there is something radically wrong with the method of proof adopted.

3. Calling the totality of the ordinary complex numbers which have α_0 for their amplitude the α_0-numbers, or the α_0 ray of numbers, Professor Miller considers the various sets of rays in regard to group properties with respect to the fundamental operations of arithmetic. In accord with this general nomenclature, the positive numbers are called 0-numbers and the negative ones are called π-numbers, the number 0 being a special 0-number. Each of two rays whose amplitudes differ by π is said to be the extension of the other, and it is observed that a ray and its extension always form a group with respect to addition or subtraction, but that no other finite number of rays has this property. The common rule for adding a positive and a negative number is included in the following: To find the sum of a number on a ray and a number on its extension, take the difference of their absolute values and prefix the angle of the one which has the larger absolute value. If a set of numbers forms a group with respect to multiplication or division, it involves either only one number from each ray represented in the set or it involves an infinite number of numbers from each one of these rays; and if a finite number of rays forms a group with respect to these operations, it includes the ray of 0-numbers. The 0-numbers and the π-numbers constitute the only finite number of rays forming a group with respect to both of the operations addition and multiplication, and this is one reason why these numbers are of such special importance. If a finite number of rays forms a group with respect to multiplication or division, these rays as units form a cyclic group. The paper will appear in the *American Mathematical Monthly*.

4. The character of the general solutions referred to by Professor Hoskins is illustrated by what may be called the generalized syllogism, which may be stated as follows: Given any two propositions involving x, y and y, z respectively, it is required to infer a proposition involving x, z. The three propositions being written in the most general equational form, according to Boole's symbolism, the result of the solution is so to express the coefficients that when the premises are given it is seen at a glance which (if any) of the coefficients in the conclusion are determined. The method of solution consists in regarding each of the three propositions forming the premises and the conclusion as derived by elimination from the general proposition involving x, y, z. The general result includes both universal and particular (or existential) propositions.

5. This paper defines by means of certain inequalities the manner in which the development in a continued fraction may be formed. These inequalities lead to three distinct types. Dr. McDonald has investigated the rapidity of convergence of each type.

6. Mr. Suter discussed certain properties of the surface $F_1(u) = u/(1 - u^4)$, $F_2(v) = v/(1 - v^4)$ by means of a system of equations of the form

$$(a)\ x = \phi_1(\xi, \eta), \quad (b)\ y = \phi_2(\xi, \eta), \quad (c)\ z = \phi_3(\xi, \eta),$$

found by substituting the given functions in Weierstrass's formulas for expressing the rectangular coordinates of any point on a minimal surface in terms of the imaginary parameters $u = \xi + i\eta$, $v = \xi - i\eta$. He showed that every pair of values (ξ, η) which satisfies the hyperbola $\eta^2 - 2\xi\eta\alpha^{-1} - \xi^2 + 1 = 0$, gives by substitution in (a) and (c), points on the section of the surface with the plane $y = -\frac{1}{2}\tan^{-1}\alpha$; and suggested in this manner a method for modelling the surface; namely, by constructing sections in planes parallel to the xz plane.

7. Only a finite number of types of trigonometries are possible under the following set of assumptions made in regard to a system of an infinite number of points (which we call a plane):

(1) With every pair of points a, b of the system belong three numbers; one we call the distance, (ab) or (ba); and the other two directions, $\overline{ab}$ and $\overline{ba}$ respectively.

(2) The existence of a trigonometry is assumed. If a, b, c are any three points of the system, we postulate a relation of the form

$$\overline{ba} - \overline{bc} = f[(ab), (bc), (ac)],$$

where $f(\alpha, \beta, \gamma)$ satisfies certain conditions as to continuity, etc., when α, β, γ are considered as independent variables.

(3) If c is a constant, and α_1, α_2, β_1, β_2 independent variables, the equations

$$f(\alpha_1, \beta_1, \gamma) + f(\beta_1, c, \beta_2) + f(c, \alpha_1, \alpha_2) = 0,$$
$$f(\alpha_2, \beta_2, \gamma) + f(\beta_2, c, \beta_1) + f(c, \alpha_2, \alpha_1) = 0,$$

can be solved for γ and the solutions are identical.

Professor Blichfeldt remarked that the conditions (2) and (3) have evident geometric meanings in the ordinary euclidean and non-euclidean planes.

8. The representation proposed by Professor Hoskins depends upon an analogy which may be explained by reference to the case of three class terms x, y, z. Let the three pairs of opposite faces of a cube be regarded as representing (x, x'), (y, y'), (z, z') respectively (x' being written for not-x, etc.); then the domain common to any two of these six classes may be represented by the edge determined by the intersection of their representative planes, while the point of intersection of any three of the six planes may represent one of the eight subclasses xyz, xyz', $\cdots$. If each face of the cube is marked with the corresponding class symbol, the relations among the subclasses are seen at a glance. A convenient plane diagram is obtained by drawing a projection of a cube, with a circular space at each vertex which in any particular case can be marked in accordance with the import of a given proposition. A symmetric representation for the case of n primary class terms would require space of n dimensions. But diagrams for practical use may be obtained by repeating the diagram for three terms. Thus the case of six terms would be represented by eight similar cubes, each at a vertex of a larger cube.

W. A. MANNING,
Secretary of the Section.

A FUNDAMENTAL INVARIANT OF THE DISCONTINUOUS ζ-GROUPS DEFINED BY THE NORMAL CURVES OF ORDER n IN A SPACE OF n DIMENSIONS.

BY PROFESSOR J. W. YOUNG.

(Read before the American Mathematical Society, February 29, 1908.)

A NORMAL curve C_n of order n in a space S_n of n dimensions is transformed into itself by ∞^3 collineations

$$z_i' = \sum_{k=0}^{n} a_{ik} z_k \qquad (i = 0, 1, \cdots, n)$$

in S_n. Each of these collineations subjects the parameter $\zeta_1 : \zeta_2$ of C_n to a linear substitution $\zeta_1' = \alpha\zeta_1 + \beta\zeta_2$, $\zeta_2' = \gamma\zeta_1 + \delta\zeta_2$. The group of collineations obtained by restricting the coefficients a_{ik} to rational integral values with determinant $|a_{ik}| = 1$ is properly discontinuous, and the corresponding group Γ_n of linear fractional substitutions

$$(1) \qquad \zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} \qquad (\alpha\delta - \beta\gamma = 1),$$

on the parameter $\zeta = \zeta_1 : \zeta_2$ is hence likewise properly discontinuous. This latter group Γ_n we call the discontinuous ζ-group defined by C_n.

The arithmetic definition of discontinuous groups of substitutions (1) forms one of the fundamental problems in the theory of automorphic functions. The above is an outline of one of the few effective methods that have been suggested for this purpose;[*] but on account of its complexity little has been done with it for values of $n > 2$.[†] General results applying to any value of n are almost totally lacking. The case $n = 2$, however, has been exhaustively treated on the arithmetic side by Fricke,[‡] and has yielded results of great importance. Fundamental in Fricke's treatment of the groups Γ_2 is an invariant $J = (\alpha + \delta)^2 - 1$, from the consideration of which follows that any group Γ_2 contains no elliptic substitutions of periods other than 2, 3, 4, and 6.* It is the object of this paper to calculate the corresponding invariant for the groups Γ_n in general, and to derive the corresponding general theorem concerning the existence of elliptic substitutions in any Γ_n.

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[*] Fricke, Chicago Congress Papers, p. 85, and J. W. Young, "On a class of discontinuous ζ-groups, etc.," *Rendiconti del Circolo matematico di Palermo*, vol. 23 (1907), p. 97.

[†] The case $n = 4$ has been partially treated by the author, loc. cit.

[‡] Fricke, "Ueber indefinite Formen mit drei und vier Variabeln," *Göttinger Nachrichten*, Dec. 13, 1883. Cf. also Fricke-Klein, "Theorie der automorphen Functionen" (Leipzig, 1897), vol. 1, p. 502 ff.

The equations of any normal curve C_n can by linear transformation be reduced to the form $z_i = \zeta_1^{n-i}\zeta_2^i$ $(i = 0, 1, \cdots, n)$. We proceed first to derive the most general collineation in S_n which leaves this curve invariant. It is readily obtained by subjecting the parameter $\zeta_1 : \zeta_2$ of the curve to the substitution

$$(1') \qquad \zeta_1' = \alpha\zeta_1 + \beta\zeta_2, \quad \zeta_2' = \gamma\zeta_1 + \delta\zeta_2 \quad (\alpha\delta - \beta\gamma = 1).$$

This gives

$$z_i' = (\alpha\zeta_1 + \beta\zeta_2)^{n-i}(\gamma\zeta_1 + \delta\zeta_2)^i \quad (i = 0, 1, \cdots, n).$$

If the right-hand member be expanded and the products $\zeta_1^\lambda\zeta_2^\mu$ replaced by their equivalent z_j's, the desired collineation is obtained in the form

$$(2) \quad z_i' = \sum_{k=0}^{n} a_{ik}z_k = \sum_{j=0}^{n-i}\sum_{j'=0}^{i}\binom{n-i}{j}\binom{i}{j'}\alpha^{n-i-j}\beta^{j}\gamma^{i-j'}\delta^{j'}z_{j+j'}.$$

The collineation which transforms any C_n whatever into itself is then obtained from this by linear transformation, since all C_n's are projectively equivalent. However, it is well-known that under every such transformation the sum Σa_{ii} of the leading diagonal elements of the matrix (a_{ik}) is an absolute invariant. This invariant, which we will denote by J, turns out to be a polynomial in $\alpha + \delta$ with integral coefficients. It may be calculated as follows:

From (2) we have at once, since a_{ii} is the coefficient of z_i,

$$J = \sum_{i=0}^{n} a_{ii} = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n-i}{j}\binom{i}{i-j}\alpha^{n-i-j}\delta^{i-j}\beta^{j}\gamma^{j}.$$

If in this expression we replace $\beta\gamma$ by $\alpha\delta - 1$, we obtain

$$J = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n-i}{j}\binom{i}{i-j}\alpha^{n-i-j}\delta^{i-j}\sum_{\nu=0}^{j}(-1)^{\nu}\binom{j}{\nu}\alpha^{j-\nu}\delta^{j-\nu}$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{n-i}\sum_{\nu=0}^{j}(-1)^{\nu}\binom{n-i}{j}\binom{i}{i-j}\binom{j}{\nu}\alpha^{n-i-\nu}\delta^{i-\nu}.$$

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[*] Fricke-Klein, loc. cit., p. 517.

Now we readily verify the following relations between binomial coefficients:

$$(3)\qquad \binom{p}{j}\binom{j}{q} = \binom{p}{q}\binom{p-q}{j-q} = \binom{p}{q}\binom{p-q}{p-j}.$$

The first of these gives

$$\binom{n-i}{j}\binom{j}{\nu} = \binom{n-i}{\nu}\binom{n-i-\nu}{j-\nu},$$

whence substituting we have

$$J = \sum_i \sum_j \sum_\nu (-1)^\nu \binom{n-i}{\nu}\binom{i}{i-j}\binom{n-i-\nu}{j-\nu}\alpha^{n-i-\nu}\delta^{i-\nu}.$$

Here and in all that follows, the summations are to be extended over all possible combinations of the symbols involved. The order in which the summations are carried out is clearly immaterial. In the last expression for J we replace $\binom{i}{i-j}$ by $\binom{i}{j}$ and j by $\nu + j'$ and sum first with respect to j', making use of the formula*

$$\sum_{s=0} \binom{p}{n+s}\binom{q}{s} = \binom{p+q}{n+q}.$$

We thus obtain

$$J = \sum_i \sum_\nu \sum_{j'} (-1)^\nu \binom{n-i}{\nu}\binom{i}{\nu+j'}\binom{n-i-\nu}{j'}\alpha^{n-i-\nu}\delta^{i-\nu}$$

$$= \sum_i \sum_\nu (-1)^\nu \binom{n-i}{\nu}\binom{n-\nu}{n-i}\alpha^{n-i-\nu}\delta^{i-\nu}.$$

By the second part of (3), this may be written

$$J = \sum_\nu \sum_i (-1)^\nu \binom{n-\nu}{\nu}\binom{n-2\nu}{i-\nu}\alpha^{n-i-\nu}\delta^{i-\nu},$$

which is clearly equivalent to

$$(4)\qquad J = \sum_{\nu=0} (-1)^\nu \binom{n-\nu}{\nu}(\alpha+\delta)^{n-2\nu}.$$

This is the desired invariant. For the value $n = 2$ it evidently reduces to the expression already referred to for this case. The result obtained may be stated as follows:

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* Netto, Lehrbuch der Combinatorik, Leipzig, 1901, p. 250.

THEOREM 1. *If the parameter ζ of any normal curve of order n in a space S_n of n dimensions is transformed by a substitution* (1), *the sum J of the elements in the leading diagonal of the matrix of the corresponding collineation in S_n is independent of the curve and is given by* (4).

This yields at once a condition on the periods of the elliptic substitutions in any discontinuous group Γ_n obtained by restricting the coefficients in the collineations to rational integral values. For every substitution (1) of any such group Γ_n, *the number J must be an integer.* Now, if ω be the period of any elliptic substitution (1) in a Γ_n we have*

$$(5) \qquad \pm(\alpha + \delta) = 2 \cos \frac{\pi}{\omega} \cdot$$

If this expression be substituted in (4), remembering that J is an integer, we obtain an equation which ω must satisfy. This may be put into a more convenient form by placing $x = e^{\pi i/\omega}$, and hence placing $(\alpha + \delta) = x + x^{-1}$.† We obtain readily

$$J = \sum_{\nu}\sum_{\mu}(-1)^{\nu}\binom{n-\nu}{\nu}\binom{n-2\nu}{\mu}x^{n-2\nu-2\mu},$$

or placing $\nu + \mu = p$, we obtain the coefficient of x^{n-2p} in the form

$$\sum_{\nu}(-1)^{\nu}\binom{n-\nu}{\nu}\binom{n-2\nu}{p-\nu}\cdot$$

By applying the first relation (3) in the opposite direction, we obtain

$$\binom{n-\nu}{\nu}\binom{n-2\nu}{p-\nu} = \binom{n-\nu}{p}\binom{p}{\nu}\cdot$$

But by a well-known relation we have‡

$$\sum_{\nu}(-1)^{\nu}\binom{n-\nu}{p}\binom{p}{\nu} = \binom{n-p}{n-p} = 1.$$

We have therefore the relation

$$J = x^{n} + x^{n-2} + x^{n-4} + \cdots + x^{-n+2} + x^{-n},$$

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* Fricke-Klein, loc. cit., p. 517.

† The signs of α, δ can always be so chosen that the upper sign in (5) holds.

‡ Netto, loc. cit., p. 252.

or multiplying by x^n,

$$x^{2n} + x^{2n-2} + x^{2n-4} + \cdots + x^2 + 1 = Jx^n.$$

Since $x^2 \neq 1$, this may also be written

$$x^{2n+2} - 1 = Jx^n(x^2 - 1). \tag{6}$$

A still more convenient form of this equation, for some purposes, may be obtained by placing $x = e^{\pi i/\omega} = \cos\omega + i\sin\omega$ and equating real and imaginary parts. By means of elementary transformations (6) is then readily shown to be equivalent to the equation

$$\sin\frac{(n+1)\pi}{\omega} = J\sin\frac{\pi}{\omega}. \tag{7}$$

These results may be stated as follows:

THEOREM 2. *In any discontinuous ζ-group defined by a normal curve C_n in a space S_n the period ω of any elliptic substitution must satisfy an equation* (7), *where J is an integer, or* (6), *where $x = e^{\pi i/\omega}$ and J is an integer.*

The numbers 2, 3 are possible periods for every value of n, as may be easily verified from the above condition. This follows also at once from the fact that the elliptic modular group, which contains substitutions of periods 2 and 3, is a Γ_n for every value of n. In fact it is the group defined by the canonical C_n which we used above to obtain the desired invariant.§ It has already been pointed out that the only other possible periods in the case $n = 2$ are 4 and 6. As examples of the application of the last theorem we give in the table below the possible periods for the values of n from 3 to 6 inclusive, calculated from (6):

Possible periods.

$n = 3$ 2, 3, 4, 5;
$n = 4$ 2, 3, 4, 5, 6;
$n = 5$ 2, 3, 5, 6, 7;
$n = 6$ 2, 3, 4, 6, 7, 8.

Moreover, by placing $J = 0$ and ± 1, it is readily seen that every factor of n, $n + 1$, and $n + 2$ is a possible period in any Γ_n.

PRINCETON UNIVERSITY,
February, 1908.

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§ Cf. Young, loc. cit., p. 99, footnote.

ON CERTAIN CONSTANTS ANALOGOUS TO FOURIER'S CONSTANTS.

BY MR. CHARLES N. MOORE.

(Read before the American Mathematical Society, April 25, 1908.)

IN the course of an article which appeared recently in the *Rendiconti del Circolo Matematico di Palermo*,[*] Landau has reproduced two proofs of the following theorem:

A. If the function $\psi(x)$ *is continuous in the interval* $(0 \leqq x \leqq 1)$ *and if*

$$(1) \qquad \int_0^1 x^\nu \psi(x) dx = 0 \qquad (\nu = 0, 1, 2, \cdots),$$

then

$$\psi(x) = 0 \qquad (0 \leqq x \leqq 1).$$

The proofs that Landau gives in detail are due to Lerch and Stieltjes. In addition he cites a second proof due to Stieltjes and a proof due to Phragmen.

As far as I am able to learn, no one seems to have mentioned the fact that this theorem, of which so many proofs have been given, is essentially equivalent to a theorem due to Hurwitz[†] which may be stated as follows:

B. If in the interval $(0 \leqq x \leqq 2\pi)$ *the function* $f(x)$ *is finite and integrable and if all of its Fourier's constants are zero, then* $f(x)$ *is zero at every point of the interval at which it is continuous.*

Theorem (A) may be deduced from (B) as follows:

It is obvious that if $\psi(x)$ is finite and integrable in the interval $(0 \leqq x \leqq 1)$ and if condition (1) is fulfilled, then the function

$$f(y) = \psi(y/2\pi)$$

satisfies all the conditions of Hurwitz's theorem. For

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[*] Vol. 25 (1908), p. 1.

[†] Cf. *Mathematische Annalen*, vol. 57 (1903), p. 440. Cf. also Bonnet, *Mémoires de l'Académie de Belgique*, vol. 23 (1850), p. 11.

$$\frac{1}{\pi}\int_0^{2\pi} f(y)\cos ny dy = 2\int_0^1 \psi(x)\cos 2n\pi x dx$$

$$= 2\left[\int_0^1 \psi(x)dx - \frac{4n^2\pi^2}{2\,!}\int_0^1 x^2\psi(x)dx + \cdots\right] = 0\,*,$$

and similarly

$$\frac{1}{\pi}\int_0^{2\pi} f(y)\sin ny dy = 0.$$

Consequently $f(y)$ is zero at every point of the interval $(0 \leqq y \leqq 2\pi)$ at which it is continuous, and hence $\psi(x)$ is zero at every point of the interval $(0 \leqq x \leqq 1)$ at which it is continuous.

Thus Hurwitz's theorem gives us an immediate proof of theorem (A) in the more general case in which $\psi(x)$ is merely finite and integrable. However, as Professor Bôcher has pointed out,† Hurwitz's theorem holds even when $f(x)$ becomes infinite at a finite number of points, provided

$$\int_0^{2\pi} |f(x)|\,dx$$

converges. Therefore theorem (A) holds for the case in which

$$\int_0^1 |\psi(x)|\,dx$$

is convergent.‡

I will now give a rather simple proof of theorem (A) for a still more general case and will then obtain theorem (B) under equally general conditions, as a consequence of this theorem. As far as I know, neither theorem has been proved before with the same degree of generality.

I will begin by proving two lemmas.

LEMMA 1. *If* $\psi(x)$ *is continuous in the interval* $(0 \leqq a \leqq x \leqq b)$ *and if*

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* We have a right to multiply the series for the cosine by a function that is finite and integrable and to integrate it term by term since the series is uniformly convergent throughout any finite interval.

† Cf. *Annals of Mathematics*, vol. 7 (1906), p. 101.

‡ This is in view of the fact that we have a right to multiply the series for the cosine by a function that is absolutely integrable and to integrate term by term.

$$(2) \qquad \int_a^b x^{\gamma+n}\psi(x)dx = 0 \quad (\gamma \geqq 0\,;\ n = 0, 1, 2, \cdots),$$

then

$$\psi(x) = 0 \qquad (a \leqq x \leqq b).$$

Since $\psi(x)$ is continuous, we know from a theorem due to Weierstrass* that we can develop it in a uniformly convergent series of polynomials in x so that we have

$$(3) \qquad \psi(x) = P_0(x) + P_1(x) + P_2(x) + \cdots.$$

We now multiply both sides of equation (3) by $x^\gamma\psi(x)$, and integrate from a to b. Since the series on the right hand side of (3) is uniformly convergent, we have a right to integrate term by term and hence in view of condition (2) the whole right hand side vanishes, so that we get

$$\int_a^b x^\gamma[\psi(x)]^2dx = 0.$$

Consequently, since $\psi(x)$ is continuous, it must be zero at every point of the interval $(a \leqq x \leqq b)$ and the lemma is proved.

By a change of variable this lemma can be thrown into the slightly more general form:

LEMMA 2. *If $\psi(x)$ is continuous in the interval $(0 \leqq a \leqq x \leqq b)$ and if*

$$(4) \qquad \int_a^b x^{\gamma+n\alpha}\psi(x)dx = 0 \quad (\gamma \geqq 0, \alpha > 0\,;\ n = 0, 1, 2, \cdots),$$

then

$$\psi(x) = 0 \qquad (a \leqq x \leqq b).$$

We are now in a position to prove the following two theorems:

THEOREM I. *If in the interval $(0 \leqq a \leqq x \leqq b)$ $\psi(x)$ is finite save for a finite number of points, and is integrable, and if furthermore*

$$(5) \qquad \int_a^b x^{\gamma+n\alpha}\psi(x)dx = 0\,† \quad (\gamma \geqq 0, \alpha > 0\,;\ n = 0, 1, 2, \cdots),$$

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* Cf. Picard, Traité d'Analyse, vol. 1, 2d ed., p. 279. Lerch's proof referred to above is based on a slightly different form of this theorem. The proof as I have given it, however, is considerably briefer.

† Condition (5) is not essentially more general than condition (1), since it involves merely a change of notation. The real generalization obtained in Theorem I is in the removal of restrictions upon $\psi(x)$.

then $\psi(x)$ *is zero at every point of the interval* $(a \leqq x \leqq b)$ *at which it is continuous.*

Let

$$\int_a^x x^\gamma \psi(x)dx = \chi(x). \tag{6}$$

Then $\chi(x)$ is continuous in the interval $(a \leqq x \leqq b)$ and in view of (6) and (5)

$$\chi(a) = 0, \quad \chi(b) = 0. \tag{7}$$

Let us take the integer m so great that $ma > 1$. Then for $n = m + k \geqq m$ we have from an integration by parts

$$\int_a^b x^{\gamma+na}\chi(x)dx = \left[x^{na}\chi(x)\right]_a^b - \int_a^b x^{ma-1+ka}\chi(x)dx \quad (k=0, 1, 2, \cdots)$$

or in view of (5) and (7)

$$\int_a^b x^{ma-1+ka}\chi(x)dx = 0 \qquad (k = 0, 1, 2, \cdots)$$

and hence from Lemma 2

$$\chi(x) = \int_a^x x^\gamma \psi(x)dx = 0 \qquad (a \leqq x \leqq b). \tag{8}$$

Differentiating (8), we see that $x^\gamma\psi(x)$, and consequently $\psi(x)$, is zero at every point of the interval $(a \leqq x \leqq b)$ at which it is continuous.*

THEOREM II. *If in the interval* $(0 \leqq a \leqq x \leqq b)$ $\psi(x)$ *is finite save for a finite number of points, and is integrable, and if furthermore*

$$\begin{aligned} &\int_a^q \psi(x)\cos nxdx = 0 \qquad (n = 0, 1, 2, \cdots), \\ &\int_a^b \psi(x)\sin nxdx = 0 \qquad (n = 1, 2, \cdots), \end{aligned} \tag{9}$$

then $\psi(x)$ *is zero at every point of the interval* $(a \leqq x \leqq b)$ *at which it is continuous.*

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* If $a = 0$ and $\gamma > 0$ it does not necessarily follow that, when $x^\gamma\psi(x)$ is zero for $x = 0$, $\psi(x)$ is zero also. However, when (8) is fulfilled for $a = 0$, it is easily seen from the integrability of $\psi(x)$ that it must be zero for $x = 0$ if it is continuous there.

We know that in any fixed interval x^n can be developed in a Fourier's series that is uniformly convergent throughout the interval and which when differentiated term by term will yield a series that is uniformly convergent throughout the same interval and which represents there the derivative of x^n.* Hence we have

$$x^n = a_0 + a_1 \cos x + b_1 \sin x + \cdots \quad (a \leqq x \leqq b), \tag{10}$$

$$nx^{n-1} = - a_1 \sin x + b_2 \cos x - \cdots \quad (a \leqq x \leqq b). \tag{11}$$

In view of (10) and the hypothesis that $\psi(x)$ is integrable in the interval $a \leqq x \leqq b$, we have

$$\begin{aligned} b^n \int_a^b \psi(x)dx = a_0 \int_a^b \psi(x)dx + a_1 \cos b \int_a^b \psi(x)dx \\ + b_1 \sin b \int_a^b \psi(x)dx + \cdots. \end{aligned} \tag{12}$$

Now let

$$\int_a^x \psi(x)dx = \chi(x). \tag{13}$$

Then $\chi(x)$ is continuous in the interval $(a \leqq x \leqq b)$, and since the series on the right hand side of (11) is uniformly convergent, we have

$$\begin{aligned} - n \int_a^b x^{n-1}\chi(x)dx = a_1 \int_a^b \chi(x) \sin x dx \\ - b_1 \int_a^b \chi(x) \cos x dx + \cdots. \end{aligned} \tag{14}$$

Adding (12) and (14), we get

$$\begin{aligned} & b^n \int_a^b \psi(x)dx - n \int_a^b x^{n-1}\chi(x)dx \\ & \quad = a_0 \int_a^b \psi(x)dx + a_1 \left[\cos b \int_a^b \psi(x)dx + \int_a^b \chi(x) \sin x dx \right] \\ & \qquad + b_1 \left[\sin b \int_a^b \psi(x)dx - \int_a^b \chi(x) \cos x dx \right] + \cdots \end{aligned}$$

or

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* For sufficient conditions that a function may be developed in a uniformly convergent Fourier's series and that a uniformly convergent Fourier's series representing its derivative may be obtained by differentiating that series term by term see Professor Bôcher's article referred to above.

$$\int_a^b x^n\psi(x)dx = a_0\int_a^b \psi(x)dx + a_1\int_a^b \psi(x)\cos x dx$$
$$+ b_1\int_a^b \psi(x)\sin x dx + \cdots = 0 \quad (n = 1, 2,\cdots).$$

Hence $\psi(x)$ satisfies the conditions of Theorem I, and therefore it is zero at every point at which it is continuous.

The method of proof used in Theorem II may be applied in the case of developments in terms of any other normal functions, such as Bessel functions, Legendre's polynomials, etc., whenever we know that x^n can be developed in a convergent series of such functions which, when differentiated term by term, will yield a uniformly convergent series that represents the derivative of x^n.* The method will enable us to show in such cases that if the coefficients of the development corresponding to any function which we know to be finite, save for a finite number of points, and integrable, are all zero, the function is zero at every point at which it is continuous.

NOTE ON THE SECOND VARIATION IN AN ISOPERIMETRIC PROBLEM.

BY DR. ELIJAH SWIFT.

(Read before the American Mathematical Society, April 25, 1908.)

SUPPOSE we have before us the simplest type of isoperimetric problem, namely to determine x and y as functions of a parameter t, so that the definite integral

$$J = \int_{t_0}^{t_1} F(x, y, x', y')dt$$

shall be a minimum, while another definite integral

$$K = \int_{t_0}^{t_1} G(x, y, x', y')dt$$

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* The existence of such developments can be proved for some of these cases by means of some theorems discussed by Stekloff. Cf. *Mémoires de l'Académie de St. Pétersbourg*, ser. 8, vol. 15 (1904).

takes a given value l, the functions $x(t)$ and $y(t)$ being subject to the condition that they take given values x_0, y_0 and x_1, y_1 for $t = t_0$ and $t = t_1$ respectively.

The object of this note is to point out a way of obtaining the second variation of J that seems to me clearer than that found in most text-books,* and which supplies the reasons for the usual derivation.

If we have a solution of the problem, $x = x(t)$, $y = y(t)$, we vary the integral J by replacing x and y by $x + \xi$ and $y + \eta$ respectively, where for all variations that leave K unchanged ξ and η may be expressed as power series in a quantity ϵ_1; if for convenience sake we assume that K and F are analytic, and taking ϵ_1 as principal infinitesimal, are in general of the first order.† We thus obtain a new integral $J + \Delta J$. The first variation of J consists of all terms of ΔJ which are of the first order; the second variation, of all those of the second order, and so on.

To obtain the second variation Bolza proceeds as follows: ‡ Since for all admissible variations K must remain equal to l, we must have $\Delta K = 0$. ΔJ is therefore unchanged if we add any multiple of ΔK to it, in particular $\Delta J = \Delta J + \lambda_0 \Delta K$, where λ_0 is the isoperimetric constant.§ Developing this expression and taking the terms of the second degree in ξ, η and their derivatives we have the usual expression for the second variation $\Delta^2 J$.

We naturally ask: Why add ΔK at all to ΔJ? And why add exactly $\lambda_0 \Delta K$? Moreover if $\Delta K = 0$, how does it happen that adding it to ΔJ makes any difference at all in the second variation? In considering a special isoperimetric problem ‖ I was led to attempt to answer these questions.

The total variation of J is

$$\Delta J = \int_{t_0}^{t_1} \left[\left(\frac{\partial F}{\partial x} \xi + \frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial x'} \xi' + \frac{\partial F}{\partial y'} \eta' \right) + \frac{1}{2} \left(\frac{\partial^2 F}{\partial x^2} \xi^2 + \cdots + \frac{\partial^2 F}{\partial y'^2} \eta'^2 \right) + \cdots \right] dt.$$

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* E. g., Bolza, "Lectures on the Calculus of Variations," p. 216.

† Bolza, pp. 206 ff.

‡ Bolza, loc. cit.

§ Bolza, pp. 208, 209.

‖ Dissertation, Göttingen, 1907, pp. 16 ff.

To find the second variation, we must take all terms that are of the second order. *But these come, not merely from the terms of the second degree in ξ, η and their derivatives, but also from those of the first, since ξ and η are power series in ϵ_1 and involve other powers of ϵ_1 besides the first.* We must take then all terms of the second order in ΔJ. Then we have $\Delta^2 J =$ all terms of the second order in

$$\int_{t_0}^{t_1}\left[\left(\frac{\partial F}{\partial x}\xi + \cdots + \frac{\partial F}{\partial y'}\eta'\right) + \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}\xi^2 + \cdots + \frac{\partial^2 F}{\partial y'^2}\eta'^2\right)\right] dt.$$

The question comes up — How pick out the terms of the second order in the first term in ΔJ and so obtain *an explicit expression* for the second variation?

We can do this by using the fact that since $\Delta K = 0$, all the terms of the second order in ΔK must vanish, and moreover the first variation of ΔJ is zero.[*] The two corresponding equations are $\Delta^2 K =$ all terms of the second order in

$$\int_{t_0}^{t_1}\left[\left(\frac{\partial G}{\partial x}\xi + \cdots + \frac{\partial G}{\partial y'}\eta'\right) + \frac{1}{2}\left(\frac{\partial^2 G}{\partial x^2}\xi^2 + \cdots + \frac{\partial^2 G}{\partial y'^2}\eta'^2\right)\right] dt = 0$$

and

$$\int_{t_0}^{t_1}\left[\left(\frac{\partial F}{\partial x}\xi + \cdots + \frac{\partial F}{\partial y'}\eta'\right) + \lambda_0\left(\frac{\partial G}{\partial x}\xi + \cdots + \frac{\partial G}{\partial y'}\eta'\right)\right] dt = 0.$$

It is at once evident that we can eliminate the terms of the first degree in ΔJ by adding to it $\lambda_0 \Delta K$, and so obtain an explicit expression for the second variation. If we call $F + \lambda_0 G = H$, we have $\Delta^2 J = \Delta^2 J + \lambda_0 \Delta^2 K =$ all terms of the second order in

$$\frac{1}{2}\int_{t_0}^{t_1}\left(\frac{\partial^2 H}{\partial x^2}\xi^2 + \cdots + \frac{\partial^2 H}{\partial y'^2}\eta'^2\right) dt,$$

the form in which the second variation is usually written.

The advantage of adding $\lambda_0 \Delta K$ to ΔJ is then that by so doing we obtain the terms of the second order in the isoperimetric problem explicitly from H in the same way as in the ordinary problem from F.

PRINCETON, N. J.,
February, 1908.

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[*] Bolza, p. 209.

NOTE ON A CERTAIN EQUATION INVOLVING THE FUNCTION $E(x)$.

BY PROFESSOR R. D. CARMICHAEL.

RECENTLY J. V. Pexider has studied the equation*

$$E\left(\frac{n+a}{x}\right)-E\left(\frac{n+a}{x+1}\right)=d, \tag{1}$$

where $E(s)$ is the greatest integer $\leqq s$, and where a is zero or a positive quantity less than 1, n is a known positive integer, d is zero or a known positive integer, and x is an unknown positive integer. M. Pexider confines himself chiefly to the case in which $d=0$ and x is less than n. He finds the values of x which satisfy the equation subject to these restrictions.

In the present note it is proposed to exhibit a simple working method by which the roots can be found in any case. When $d>0$, x is always less than n, except that x may equal n when $d=1$. In what follows x is taken always less than n.

If $n+a$ is divided by an integer i, giving the quotient $q+\beta$ where β is zero or a positive quantity less than 1; then if n is also divided by i, the quotient will evidently be $q+y$ where y is zero or a positive quantity less than 1. Hence

$$E\left(\frac{n+a}{i}\right)=E\left(\frac{n}{i}\right).$$

Therefore, the equation

$$E\left(\frac{n}{x}\right)-E\left(\frac{n}{x+1}\right)=d \tag{2}$$

has the same roots as (1). We may then confine ourselves to the solution of the latter equation as being somewhat the simpler of the two.

Represent n in the form

$$n=ax+c \qquad (c<x,\ a\neq 0). \tag{3}$$

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* *Rendiconti del Circolo Matem. di Palermo*, vol. 24, no. 1, pp. 46–64. For convenience I write the equation in a form somewhat different from that of M. Pexider.

Then

$$E\left(\frac{n}{x}\right) \equiv E\left(\frac{ax+c}{x}\right) = a;$$

and therefore

$$(4) \qquad E\left(\frac{n}{x+1}\right) \equiv E\left(\frac{ax+c}{x+1}\right) = a-d$$

is the necessary and sufficient condition that x is a root of (2). Now (4) will be satisfied if and only if

$$ax + c - (x+1)(a-d) < x+1 \quad \text{and} \quad \geqq 0.$$

This readily reduces to

$$(5) \qquad a - c > (d-1)(x+1) \quad \text{and} \quad \leqq d(x+1).$$

From these results we have the following working method for finding the solutions of (2) and hence of (1):

Write n in every possible way in the form

$$(3) \qquad n = ax + c \qquad (c < x, \ a \neq 0)$$

and examine whether both inequalities

$$(5) \qquad a - c > (d-1)(x+1) \quad \text{and} \quad \leqq d(x+1)$$

are fulfilled at the same time; if so, x is a root of the equation; otherwise, it is not such a root.

Applying this to the case when $d = 0$, we have to determine x subject to the conditions

$$n = ax + c, \quad (c < x, \ a \neq 0, \ c - a < x + 1 \ \text{and} \ \geqq 0).$$

M. Pexider (l. c., page 57) shows that the number of such roots (x being less than n) is

$$A(n) = E(n) - E(\sqrt{n}) - E\left(\frac{n}{E(\sqrt{n})+1}\right).$$

He exhibits also a better method than the above by which the roots may be determined in the present case.

If $d = 1$, the roots of our equation are to be found from

$$n = ax + c, \quad (c < x, \ a \neq 0, \ a - c > 0 \ \text{and} \ \leqq x + 1).$$

This will give all the roots of (1) and (2) except $x = n$ which has been definitely excluded from this result by previous assumption that $x < n$.

For other values of d the expressions in (5) do not simplify, and we have to determine x from the general conditions

$$n = ax + c,\ c < x,\ (a \neq 0,\ a - c > (d-1)(x+1) \text{ and } \leqq d(x+1)).$$

It may be verified by actual trial that the solutions of (1) and (2) may in this way be effected with a degree of readiness which will make the method serviceable in a large number of cases.

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THE INNER FORCE OF A MOVING ELECTRON.

BY DR. F. R. SHARPE.

(Read before the American Mathematical Society, February 29, 1908.)

1. *Introduction.* — Sommerfeld* has given a general method of determining the force which an electron exerts on itself when its motion is known. Schott† has applied the retarded potential to the same problem. In the present paper the known vector expression‡ for the force with which a moving point charge acts on another point charge is used to determine the action between any two elements of the electron. A double integration over the volume of the electron gives the inner force of the electron. The Abraham-Sommerfeld expressions for the longitudinal mass and Abraham's value for the transverse mass are thus very simply determined and the limitations on the solutions are made manifest.

2. *The Force Between Two Moving Point Charges.* — The electron is assumed to be a uniformly charged solid sphere which is moving in a straight line without rotation. Take the x-axis in the direction of motion. Measure the time from the instant at which the force is to be determined and choose the velocity of light as the unit of velocity. Let (x_0, y, z) be the coordinates of the point charge de_2 relative to the point charge de_1. The distance moved by the electron in time t is

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* *Göttinger Nachrichten*, 1904.

† *Ann. der. Physik*, 1908; No. 1.

‡ Abraham : Theorie der Elektrizität, vol. II, p. 98.

$$v_0 t + \tfrac{1}{2}\dot{v}_0 t^2 + \tfrac{1}{6}\ddot{v}_0 t^3 + \cdots,$$

where v_0 is its velocity at time 0. The point charge de_1 will therefore act on de_2 at the time $-\tau$ which is determined so as to satisfy

$$(1) \qquad \tau^2 = y^2 + z^2 + (x_0 + v_0\tau - \tfrac{1}{2}\dot{v}_0\tau^2 + \cdots)^2.$$

The velocity of de_1 at this time is

$$(2) \qquad v = v_0 - \dot{v}_0\tau + \tfrac{1}{2}\ddot{v}_0\tau^2 - \cdots,$$

and its acceleration is

$$(3) \qquad \dot{v} = \dot{v}_0 - \ddot{v}_0\tau + \cdots.$$

If the velocity of de_1 had remained constant and equal to v from the time $-\tau$, its coordinates at time 0 would have been (x, y, z), where

$$x = x_0 + v_0\tau - \tfrac{1}{2}\ddot{v}_0\tau^2 + \cdots - \tau(v_0 - \dot{v}_0\tau + \cdots)$$

or

$$(4) \qquad x = x_0 + \tfrac{1}{2}\dot{v}_0\tau^2 + \cdots.$$

The x component of the force of de_1 on de_2 can be reduced from its vector form to

$$(5) \qquad \frac{x(1 - v^2) - \dot{v}(y^2 + z^2)}{\{x^2 + (1 - v^2)(y^2 + z^2)\}^{\frac{3}{2}}}\, de_1 de_2.$$

The effect of de_2 on de_1 is obtained by changing the sign of x_0.

3. *Uniform Motion.*—In this case v is v_0 and x is x_0, so that (5) becomes

$$\frac{x_0(1 - v_0^2)}{\{x_0^2 + (1 - v_0^2)(y^2 + z^2)\}^{\frac{3}{2}}}\, de_1 de_2.$$

Hence the mutual actions of de_1 and de_2 are equal and opposite and the inner force is zero.

4. *Slow Uniform Acceleration.*—Here $\dot{v}$ is $\dot{v}_0$, and the unbalanced part of the force of de_1 on de_2 is

$$-\frac{\dot{v}_0(y^2 + z^2)de_1 \cdot de_2 \cdot}{\{x_0^2 + (1 - v_0^2)(y^2 + z^2)\}^{\frac{3}{2}}},$$

which may be written

$$-\frac{\dot{v}_0\partial}{v_0\partial v_0}\left[\frac{de_1de_2}{\{x_0^2+(1-v_0^2)(y^2+z^2)\}^{\frac{1}{2}}}\right].$$

The inner force of the electron is twice the sum of this expression for every pair of elements of the electron. Let the coordinates of de_1 and de_2 referred to the center of the electron be (x_1, y_1, z_1) and (x_2, y_2, z_2). Then

$$x_0=x_2-x_1, \quad y=y_2-y_1, \quad \text{and} \quad z=z_2-z_1.$$

If we now make the transformation

$$x=\sqrt{1-v_0^2}\cdot x', \quad \rho=\frac{\rho'}{\sqrt{1-v_0^2}},$$

where ρ is the volume density of the charge e of the electron, the charge is unaltered, but the spherical electron of radius a is transformed into the prolate spheroid

$$x'^2(1-v_0^2)+y^2+z^2=a^2.$$

The expression which we have to integrate is therefore $2/\sqrt{(1-v_0^2)}$ of the self potential of this spheroid and has the value*

$$\frac{6}{5}\cdot\frac{e^2}{2av_0}\log\left(\frac{1+v_0}{1-v_0}\right).$$

Hence the inner force of the electron is

$$-\frac{6}{5}\cdot\frac{e^2\dot{v}_0}{2a}\cdot\frac{\partial}{v_0\partial v_0}\left\{\frac{1}{v_0}\log\left(\frac{1+v_0}{1-v_0}\right)\right\},$$

that is,

$$-\dot{v}_0\cdot\frac{6}{5}\frac{e^2}{2av_0^2}\left(-\frac{1}{v_0}\log\frac{1+v_0}{1-v_0}+\frac{2}{1-v_0^2}\right).$$

The coefficient of $-\dot{v}_0$ is Abraham's expression for the longitudinal mass in the case of quasi-stationary motion.

5. *Variable Acceleration.*—In the previous case the changes in the velocity and acceleration were neglected. If we retain the first power of τ, we have from (2) and (3)

$$v=v_0-\dot{v}_0\tau, \quad \dot{v}=\dot{v}_0-\ddot{v}_0\tau,$$

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* Abraham, *loc. cit.*, p. 179.

and from (1)

$$\tau = \frac{v_0 x + \sqrt{x_0^2 + (1 - v_0^2)(y^2 + z^2)}}{1 - v_0^2}.$$

Substituting in (5), we find the additional unbalanced terms

$$\frac{\ddot{v}_0(y^2 + z^2)de_1de_2}{(1 - v_0^2)\{x_0^2 + (1 - v_0^2)(y^2 + z^2)\}} + \frac{3\dot{v}_0^2 v_0(y^2 + z^2)^2 de_1 de_2}{(1 - v_0^2)\{x_0^2 + (1 - v_0^2)(y^2 + z^2)\}^2}.$$

Denoting by θ the angle which the line joining de_1 to de_2 makes with the x axis, we have

$$\frac{\ddot{v}_0}{1 - v_0^2} \cdot \frac{\sin^2 \theta de_1 de_2}{1 - v_0^2 \sin^2 \theta} + \frac{3\dot{v}_0^2}{2(1 - v_0^2)} \frac{\partial}{\partial v_0}\left(\frac{\sin^2 \theta de_1 de_2}{1 - v_0^2 \sin^2 \theta} \right).$$

Consider

$$\int\int \frac{\sin^2 \theta de_1 de_2}{1 - v_0^2 \sin^2 \theta};$$

since it is independent of the length of the line joining de_1 to de_2, its value is

$$\frac{e^2}{4\pi}\int \frac{\sin^2 \theta}{1 - v_0^2 \sin^2 \theta} dw,$$

where dw is the element $2\pi \sin \theta d\theta$ of the complete solid angle 4π. Integrating from 0 to π, we find the value of the additional terms to be

$$\frac{\ddot{v}_0 e^2}{1 - v_0^2}\left(-\frac{1}{v_0^2} + \frac{1}{v_0^3 \sqrt{1 - v_0^2}} \tan^{-1} \frac{v_0}{\sqrt{1 - v_0^2}} \right)$$
$$+ \frac{3\dot{v}_0^2 e^2}{2(1 - v_0^2)} \frac{\partial}{\partial v_0}\left(-\frac{1}{v_0^2} + \frac{1}{v_0^3 \sqrt{1 - v_0^2}} \tan^{-1} \frac{v_0}{\sqrt{1 - v_0^2}} \right).$$

This result differs somewhat from the expression found by Schott and Abraham.

6. *Transverse Acceleration.*—When the velocity is constant but there is a small transverse acceleration f, we may choose the y axis in the direction of f. The vector expression for the y component of the unbalanced force may be reduced to

$$-f\frac{x_0^2 + (1 - v_0^2)z^2}{\{x_0^2 + (1 - v_0^2)(y^2 + z^2)\}^{\frac{3}{2}}} de_1 de_2,$$

that is,

$$-f de_1 de_2 \left[\frac{1}{\{x_0^2 + (1 - v_0^2)(y^2 + z^2)\}^{\frac{1}{2}}} - \frac{(1 - v_0^2) y^2}{\{x_0^2 + (1 - v_0^2)(y^2 + z^2)\}^{\frac{3}{2}}} \right].$$

Integrating as in the previous cases, we have

$$-f \cdot \frac{6}{5} \cdot \frac{e^2}{2a} \left\{ \frac{1}{v_0} \log \frac{1 + v_0}{1 - v_0} - \frac{(1 - v_0^2)\partial}{2 v_0 \partial v_0} \left(\frac{1}{v_0} \log \frac{1 + v_0}{1 - v_0} \right) \right\}$$

or

$$-f \cdot \frac{6}{5} \cdot \frac{e^2}{2a v_0^2} \left\{ \frac{1 + v_0^2}{2 v_0} \log \frac{1 + v_0}{1 - v_0} - 1 \right\},$$

which gives Abraham's expression for the transverse mass.

CORNELL UNIVERSITY,
March, 1908.

THE RECENTLY DISCOVERED MANUSCRIPT OF ARCHIMEDES.

PROFESSOR J. L. Heiberg has published two important accounts of his recent discovery of a new manuscript of Archimedes, both of which are of great interest to mathematicians. The first of these accounts is printed in volume 42 of *Hermes*. It contains the Greek text of a lost treatise of Archimedes, which is recovered nearly complete in the newly found manuscript. A German translation of the Greek text, and an interesting commentary by Zeuthen, is printed by Heiberg in the *Bibliotheca Mathematica*, volume 7, page 321.

Professor Heiberg's critical study of Archimedes has extended over a period of more than thirty years. His dissertation, "Quæstiones Archimedeæ" (Copenhagen, 1879), is constantly referred to by students of Archimedes, both on account of its scholarly critique of Archimedes's work, and on account of the innumerable references to Archimedean literature which are there brought together for the first time. In 1880–1881 Heiberg published the definitive edition of Archimedes, with which his name is usually associated. It was while at work upon a second edition of this book that Heiberg's attention was directed to a palimpsest manuscript of mathematical content recently catalogued in a cloister at Constantinople. Failing in an attempt to have the manuscript sent to him at Copenhagen, Heiberg spent the summer of 1906 in Constantinople. Investigation soon showed that the manuscript contained important new material, the study of which would cost more time than was then at Heiberg's disposal. Consequently photography was used wherever possible to copy the pages containing the new material. A facsimile is printed on page 235 of the volume of *Hermes* referred to above.

The contents of the palimpsest are as follows: (1) Large portions of the previously known writings of Archimedes, the Sphere and Cylinder and On Spirals, nearly complete, parts of Measurement of the Circle, and Equilibrium of Planes; (2) a large part of the book On Floating Bodies, which was hitherto extant only in a Latin translation, so that numerous defects and lacunæ in this book can now be healed; (3) a hitherto unknown treatise quoted by Heron and referred to by Suidas, the title of which we will translate "A Method." This book constitutes the most important find of new material. There is also the beginning of "'Αρχιμήδους Στομαχ(ικόν)" which Heiberg says is a sort of "Chinese game," of which there is not enough remaining to patch up the meaning of the original.

The recovery of this lost work of Archimedes must be considered of the very foremost importance. Of course, the recovery of any work of so original a genius as Archimedes must be considered important. The special significance, however, of the present find consists in the fact that here Archimedes admits us to his mathematical workshop, and points out to us, not only how his discoveries were made, but how other discoveries may be made in the future by the method of research which he discloses. It is a common complaint that the Greek mathematics that has come down to us yields no trace of the process of discovery used by the investigator. The motto of the Greek school might well have been: "Results and not processes are for the public." This book of Archimedes is the one known exception, for herein he relates in charming detail the story of the discovery of some of the most fundamental and most interesting theorems of geometry.

Archimedes sends "A Method" to Eratosthenes, in order to make him acquainted with his method of discovery. It is a conscious attempt, not only to hand Eratosthenes the demonstration of certain theorems which Archimedes had propounded to him, without proof; but to give to Eratosthenes the knowledge of a method of research by which many new truths might be expected to be discovered. At this point it is probably best to let Archimedes speak for himself. It is a matter of great interest that the letters which Archimedes prefixed to his different mathematical treatises have come down to us. These are all of very great value in showing us the personal side of his work and in coordinating the different lines of inquiry that absorbed his attention. The present book is no exception, and the letter to Eratosthenes which opens the book is preserved intact. It reads as follows:

"*ARCHIMEDES TO ERATOSTHENES, GREETING:* I formerly sent you some theorems discovered by myself with the request that you find the proofs which I provisionally withheld. The theorems which I sent to you were as follows:

1. If a cylinder be inscribed in a rectangular prism with square base, so that its bases lie in the opposite squares, and so that elements touch the remaining planes of the prism, and if a plane be passed through the center of the circle which is the base of the cylinder and a side of the square lying in the opposite face of the prism, then this plane cuts off a segment of the cylinder which is bounded by two planes (the cutting plane and the plane wherein lies the base of the cylinder) and by the portion of the cylindrical surface lying between the planes just mentioned; this segment of the cylinder is $\frac{1}{6}$ of the whole prism.
2. If there be inscribed in a cube a cylinder which has its bases in opposite squares and touches the other four faces of the cube, and, moreover in the same cube a second cylinder be inscribed which has its bases in two other squares, and touches the other four faces of the cube, the solid enclosed by the cylindrical surfaces (which solid is common to the two cylinders), is equal to $\frac{2}{3}$ of the entire cube.

"These theorems are essentially different from those communicated by me earlier; the former bodies, the conoids and spheroids and their segments, we compared with the volumes of cones and cylinders, but none of them were found to be equal to a body bounded by planes; on the other hand, each of the present solids, bounded by two planes and cylindrical surfaces, was found equal to solids bounded by planes. I send the proof of these theorems in this book.

"However, since I see that you are an excellent scholar, and that you are not only a prominent student of philosophy, but also a lover of mathematical research, I have deemed it well to explain to you and put down in this same book a special method whereby the possibility will be offered you to investigate any mathematical question by means of mechanics. And I am convinced that it is quite as useful in the proof of the theorems themselves; for many of them that were formerly only clear to me through mechanics were afterwards proved by geometry, although the treatment by the former method was not founded on demonstration: indeed, if one has previously gotten a conception of the problem by this method, it is easier to produce the proof than to find it without a provisional conception. Likewise, if one considers the known theorems whose demonstrations were first found by Eudoxus, namely that the cone and the pyramid are respectively one third of the cylinder and prism having the same base and equal altitudes, one must award no slight credit to Democritus who first obtained expression for these truths without a proof. We are also in the position of having previously discovered in the same way the theorems now to be published, and we feel obliged to make the method known, partly because we have previously spoken of them, lest it be believed that we have spread an idle tale, partly in the conviction that there will be instituted thereby a matter of no slight utility in mathematics. In fact, I believe that living or future investigators will discover other theorems which have not occurred to us by the method here discussed.

"We will first explain a theorem that first became clear to me by means of mechanics — to wit that a segment of a parabola is $\frac{4}{3}$ the triangle that has the same base and equal altitude, but thereafter will be given in turn the several theorems found by the method described; and at the conclusion of the book we present the geometrical proof of the theorems above mentioned."

After this introduction, Archimedes gives a list of eight lemmas which he will use in the following demonstrations. All of these are theorems concerning the position of the center of gravity of known geometrical figures. The first five are found in his treatise "On the equilibrium of planes." No. 6, "The center of gravity of a circle is its center"; No. 7, "The center of gravity of a cylinder is the middle point of its axis"; No. 8, "The center of gravity of a cone divides the axis so that the portion adjacent to the apex is thrice the portion adjacent to the base" are not found in any of his existing works. They may be contained in a lost work on centers of gravity, referred to by Simplicius, for Archimedes states that these first eight have already been published. He then states another preliminary theorem, which is the same as "On conoids," 1.

The new method of Archimedes is nearly identical with the method of the integral calculus. Many of the processes used by the ancients had the semblance of the method of the infinitesimal calculus, but the present manuscript shows that Archimedes had taken the decisive step in his method of investigation and had done it consciously. In this work he asserts that surfaces are to be considered as made up of lines, and that solids of revolution are "filled up" by circles. It is true that he expressly admits that his method is not demonstration, but he is thoroughly convinced that the results obtained by it are correct. One must emphasize the fact that the conception and use of the infinite was prohibited in Greek mathematics. The paradoxes of Zeno and the sophists had overfrightened the mathematicians of the Greek school and they had found their only safety in their ingenious theory of proportion and the method of exhaustions. It is an exceedingly interesting fact that Archimedes dared to work with the new idea in his preliminary investigations and it is even more remarkable that he was willing to communicate it to his fellow workers. How very strict the mathematical traditions were on this point is well illustrated by the fact that Archimedes not only gives a geometrical proof for the volume of the segment of a cylinder in the form of the infinitesimal method suggested by his mechanical proof, but he afterwards adds a demonstration in the precise form of a regular exhaustion proof, in order that no one could question its validity.*

Archimedes has abundantly shown that he possessed the genius necessary to formulate the infinitesimal method on a sound basis. He evidently made no attempt of this sort, and it does not appear certain whether his lack of effort in this direction was due to the fact that he saw no means of avoiding in all cases the use of the principles of mechanics in applying the infinitesimal method, and hence the impossibility of a demonstration purely geometrical; or whether he was deterred from such effort on account of adhesion to the traditions of the Greek school. Besides these considerations, there is the strong native desire of great constructive intellects to leave their investigations in the synthetic form — so strongly apparent in the case of Newton and by no means extinct in later days. My own conviction, after some years of familiarity with Archimedes's writings, is that Archimedes was not willing to rest a geometrical demonstration upon mechanical principles, except provisionally, and that his reason was not that rigor could not be secured in that direction, but that there was a scientific propriety, as well as an æsthetic obligation, to keep geometry sufficient unto itself. We all see how such a doctrine fits into the platonic philosophy of Archimedes's time and is in accord with the spirit of the Alexandrian school.

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* These comments are substantially those of Heiberg. For many references given by Heiberg see *Hermes*, l. c., p. 302. The last sentence of above paragraph refers to theorems XI, XII, XIII of Archimedes's text.

Few realize, except those who have put some study upon Archimedes's work, the tremendously exacting labor that Archimedes put upon himself in placing the demonstrations of his discoveries in the form of exhaustion proofs. To him geometry was the noblest of the sciences. Plutarch may not be exaggerating when he says, "he placed his whole delight in these intellectual speculations, which, without any relation to the necessities of life, have an intrinsic excellence arising from truth and demonstration only."

The first two theorems in the new manuscript of Archimedes suffice to illustrate his method. I have abbreviated or omitted descriptive portions of the demonstrations, but have put in his exact language as far as practicable those portions which are of the greatest importance in conveying his exact point of view.

"Theorem I. Let $\alpha\beta\gamma$ be a segment of a parabola bounded by the straight line $\alpha\gamma$ and the parabola $\alpha\beta\gamma$. Let $\alpha\gamma$ be bisected at δ and let $\delta\beta\epsilon$ be drawn parallel to the diameter of the parabola. Draw $\alpha\beta$ and $\beta\gamma$. Then the segment $\alpha\beta\gamma$ is $\frac{4}{3}$ the triangle $\alpha\beta\gamma$."

"Draw from the points α, γ the line $\alpha\zeta$ parallel to $\delta\beta\epsilon$ and the tangent $\gamma\zeta$. Produce $\gamma\beta$ to κ and make $\kappa\theta = \gamma\kappa$. We then suppose that $\gamma\theta$ is a 'neck yoke'* with middle point κ

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* The Greek ζυγόν literally means yoke. It is the same word as the Latin *jugum* or English *yoke*. It came to be used for beam of a balance and Archimedes uses it as the equivalent of *lever*, by which it is rendered in translations of Archimedes.

and that $\mu\xi$ is any straight line parallel to $\epsilon\delta$. Since $\gamma\beta\alpha$ is a parabola, $\gamma\zeta$ a tangent and $\gamma\delta$ an ordinate, then $\epsilon\beta = \beta\delta$; in fact this is proved in the Elements. For this reason, and since $\zeta\alpha$ and $\mu\xi$ are parallel to $\epsilon\delta$, we know that $\mu\nu = \nu\xi$, $\zeta\kappa = \kappa\alpha$. Since $\gamma\alpha : \alpha\xi = \mu\xi : \xi o$ (in fact this was proved in a lemma [Quad. Parab. 5]), we have $\gamma\alpha : \alpha\xi = \gamma\kappa : \kappa\nu$, and since $\gamma\kappa = \kappa\theta$, therefore $\theta\kappa : \kappa\nu = \mu\xi : \xi o$. Now ν is the center of gravity of the straight line $\mu\xi$, since $\mu\nu = \nu\xi$; hence if we construct $\tau\eta = \xi o$ with θ as its center of gravity, so that $\tau\theta = \theta\eta$, then the straight line $\tau\theta\eta$ must be in equilibrium about κ with $\mu\xi$ in the position where it is; for $\theta\nu$ is divided in the inverse ratio of the weights $\tau\eta$ and $\mu\xi$ and $\theta\kappa : \kappa\nu = \mu\xi : \eta\tau$; therefore κ is the center of gravity of the combined weights. In the same way all lines which are drawn in the triangle $\zeta\alpha\gamma$ parallel to $\epsilon\delta$ are in equilibrium, in the position where they are, with their segments intercepted by the parabola, provided these latter be displaced to θ, so that κ will be the center of gravity of the combined weights. Since the lines of the triangle $\gamma\zeta\alpha$ make up the triangle, and since the lines corresponding to ξo make up the segment of the parabola $\alpha\beta\gamma$, therefore the triangle $\zeta\alpha\gamma$, in the place where it is, is in equilibrium about the point κ, with the segment of the parabola, if this latter be displaced so as to have θ as center of gravity; hence κ is the center of gravity of the combined weights.

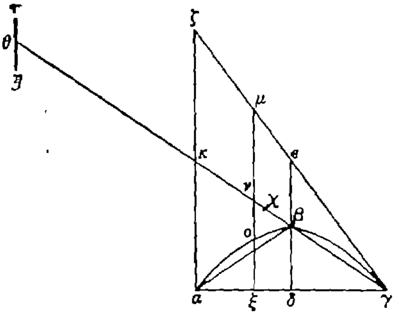

FIG. 1.

"Now let $\gamma\kappa$ be so divided at χ that $\gamma\kappa = 3\kappa\chi$; then χ is the center of gravity of the triangle $\alpha\zeta\gamma$; for this is proved in the theory of equilibrium (De plan. æquil. I, 14). Now the triangle $\zeta\alpha\gamma$, in the place where it is, is in equilibrium about κ

with the segment $\beta\alpha\gamma$, if this be displaced so that θ is its center of gravity, and the center of gravity of the triangle $\zeta\alpha\gamma$ is χ; therefore the triangle $\alpha\zeta\gamma$: segment $\alpha\beta\gamma$ displaced to θ as center of gravity :: $\theta\kappa : \kappa\chi$. But $\theta\kappa = 3\kappa\chi$; therefore the triangle $\alpha\zeta\gamma = 3$ segments $\alpha\beta\gamma$. But the triangle $\zeta\alpha\gamma = 4$ triangles $\alpha\beta\gamma$, since $\zeta\kappa = \kappa\alpha$ and $\alpha\delta = \delta\gamma$; therefore the segment $\alpha\beta\gamma = 4/3$ the triangle $\alpha\beta\gamma$."

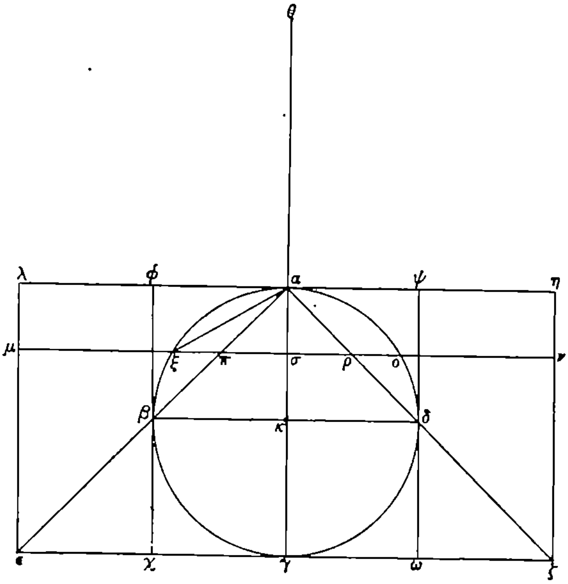

FIG. 2.

"Theorem II. That the sphere is four times as great as a cone whose base is equal to a great circle of the sphere, but whose altitude equals the radius of the sphere; and that a cylinder whose base is equal to the great circle of the sphere, but whose altitude is equal to the diameter of the circle, is one and one half times as great as the sphere, can be explained by the present method in the following manner:

"Let there be (Fig. 2) a sphere and two mutually perpendicular diameters $\alpha\gamma$, $\beta\delta$ of the great circle $\alpha\beta\gamma\delta$; there is a circle of the sphere perpendicular to $\alpha\beta\gamma\delta$ whose diameter is $\beta\delta$. On this circle let a cone be constructed whose apex is α and afterwards let its convex surface be produced until the conical surface is cut by the plane passing through γ parallel to the base of the first cone, in consequence a circle is intercepted which is perpendicular to $\alpha\gamma$ and whose diameter is $\epsilon\zeta$. Erect a cylinder on this circle whose axis is $\alpha\gamma$ and having elements $\epsilon\lambda$ and $\zeta\eta$. Produce $\gamma\alpha$ and make $\alpha\theta = \gamma\alpha$, and suppose $\gamma\theta$ to be a lever whose fulcrum is α; moreover let any line, as $\mu\nu$, be drawn parallel to $\beta\delta$; it cuts the circle $\alpha\beta\gamma\delta$ in ξ-and o, the diameter $\alpha\gamma$ in σ, the line $\alpha\epsilon$ in π and $\alpha\zeta$ in ρ. Let a plane be passed through the line $\mu\nu$ perpendicular to $\alpha\gamma$; it will cut a circle from the cylinder of diameter $\mu\nu$; it will cut the sphere in a circle of diameter ξo; it will cut the cone $\alpha\epsilon\zeta$ in a circle of diameter $\pi\rho$. Now since $\gamma\alpha \times \alpha\sigma = \mu\sigma \times \sigma\pi$ (for $\alpha\gamma = \sigma\mu$, $\alpha\sigma = \pi\sigma$) and $\gamma\alpha \times \alpha\sigma = \alpha\xi^2 = \xi\sigma^3 + \sigma\pi^2$, therefore $\mu\sigma \times \sigma\pi = \xi\sigma^2 + \sigma\pi^2$; moreover, since $\gamma\alpha : \alpha\sigma = \mu\sigma : \sigma\pi$ and $\gamma\alpha = \alpha\theta$, therefore $\theta\alpha : \alpha\sigma = \mu\sigma : \sigma\pi = \mu\sigma^2 : \mu\sigma \times \sigma\pi$. It has been shown, however, that $\xi\sigma^2 + \sigma\pi^2 = \mu\sigma \times \sigma\pi$; therefore $\alpha\theta : \alpha\sigma = \mu\sigma^2 : \xi\sigma^2 + \sigma\pi^3$. But $\mu\sigma^2 : \xi\sigma^2 + \sigma\pi^2 = \mu\nu^2 : \xi o^2 + \pi\rho^2 =$ the circle of cylinder of diameter $\mu\nu$: the circle of cone of diameter $\pi\rho$ + the circle of sphere of diameter ξo. Therefore $\theta\alpha : \alpha\sigma =$ the circle of cylinder : circle of cone + circle of sphere. Therefore, the circle of the cylinder, in the place where it is, is in equilibrium about the point α with the combined circles of diameters ξo, $\pi\rho$, if they be displaced to θ so that θ is the center of gravity of both. In the same way it can be shown that if any other line be drawn in the parallelogram $\zeta\lambda$, parallel to $\epsilon\zeta$, and a plane passed through it perpendicular to $\alpha\gamma$, then the circle intercepted by the cylinder is in equilibrium about α with the combined circles intercepted by the sphere and cone, provided they be displaced and so placed on the lever arm at θ that θ is the center of gravity of both. If, therefore, the cylinder, sphere and cone be filled up with the above mentioned circles, then the cylinder, in the place where it is, will be in equilibrium about α with the combined cone and sphere, if they be displaced and so placed on the lever arm at θ that θ is the center of gravity of both. Now since the solids named are in equilibrium, the cylinder with center of gravity κ, the cone and sphere displaced as stated with center of gravity θ, then $\theta\alpha : \alpha\kappa =$ cylinder : sphere + cone. But $\theta\alpha = 2\alpha\kappa$ and the cylinder is equivalent to three cones (Euclid, Elements XII, 10); therefore 3 cones = 2 cones + 2 spheres. If two cones be taken away from both members, the cone whose axial triangle is $\alpha\epsilon\zeta$ is equal to two spheres. But the cone whose axial triangle is $\alpha\epsilon\zeta = 8$ cones whose axial triangle is $\alpha\beta\delta$, because $\epsilon\zeta = 2\beta\delta$, therefore, the 8 cones named are equal to two spheres. Consequently the sphere whose great circle is $\alpha\beta\gamma\delta$ is four times as great as the cone whose apex is α and whose base is the circle of diameter $\beta\delta$ perpendicular to $\alpha\gamma$.

"Through β and δ draw the lines $\phi\beta\chi$ and $\psi\delta\omega$ in the prallelaogram $\lambda\zeta$ parallel to $\alpha\gamma$ and introduce the cylinder whose bases are circles having the diameters $\phi\psi$ and $\chi\omega$, and whose axis is $\alpha\gamma$. Then the cylinder whose axial parallelogram is $\phi\omega$ is twice as large as the cylinder whose axial parallelogram is $\phi\delta$, and this last is threefold as large as the cone whose axial triangle is $\alpha\beta\delta$, as is proved in the Elements. Therefore the cylinder whose axial parallelogram is $\phi\omega$ is sixfold as large as the cone whose axial triangle is $\alpha\beta\delta$. It has been proved, however, that the sphere of great circle $\alpha\beta\gamma\delta$ is fourfold as large as the same cone; consequently the cylinder is $\frac{3}{2}$ the sphere, which was to be proved.

"Because of this theorem — that a sphere is fourfold as large as the cone whose base is a great circle and whose altitude is the radius of the sphere — the thought occurred to me that the surface of the sphere is fourfold as great as its great circle, because I proceeded from the conception that as a circle is equivalent to a triangle whose base is equal to the circumference of the circle and whose altitude is equal to the radius of the circle, in the same way a sphere is equal to a cone whose base is equal to the surface of the sphere, and whose altitude is equal to the radius of the sphere."

Following the above theorems, Archimedes gives a number of others, all of which are investigated by the same method. Among these are: volume of a spheroid; volume of a paraboloid; center of gravity of a paraboloid; center of gravity of a hemisphere; volume of spherical segment; center of gravity of a spherical segment. Then follows the theorem concerning the volume of the cylindrical segment announced in the introduction. This contains a number of lacunæ. No trace is left of the theorem concerning the solid common to two intersecting cylinders, and of the geometrical proof of the area of parabolic segment, etc., which were promised in the introduction.

The two theorems reproduced above illustrate the method used by Archimedes in the remaining theorems. The method may be summarized as follows: The elements of a solid or surface to be measured are connected by a proportion with the elements of bodies previously considered, and in such manner that two terms of the proportion shall be straight lines. The straight lines may then be used as "lever arms" to balance the elements or certain combination of elements selected from the new and old bodies. Then if the magnitude and position of the centers of gravity of the old bodies be known, the new body, when displaced so that its center of gravity lies at one end of the lever arm, will be in equilibrium with a known magnitude "in the place where it is" and hence an equality is established. This means, in modern language, that a definite integral is evaluated by means of the mean value of a function defined by means of the center of gravity. On the other hand, if the magnitudes of the bodies under investigation be known, the center of gravity of one of them can be found by the same process. The method requires the introduction of the principle of moments as a postulate of mathematics and a formulation of the doctrine of infinitesimals. Archimedes avoided this by the construction of exhaustion proofs.

A number of matters of considerable historical interest are settled by the present work. It is now known that Democritus and not Eudoxus is to be given credit for the discovery of the relation between the volume of the pyramid and prism and of the cone and cylinder. Evidently Eudoxus was the first to put the demonstrations in the form of an exhaustion proof, but his work is to be compared with Archimedes's geometrical demonstrations constructed after the facts had been developed by means of mechanics. It is also of interest to note that Archimedes states that he discovered the theorem concerning the volume of the sphere previous to the discovery of the theorem expressing the area. His description of the manner in which the idea occurred to him that the surface of the sphere is fourfold its great circle, is of the very greatest interest. One would naturally but erroneously infer from the way the matter is put in his treatise on the Sphere and Cylinder, that the area of the sphere became known to him before the discovery of its volume.

Archimedes makes it clear that he is using this method to "discover" the theorems, and that it is "not founded on demonstration." But in speaking of spheres as "filled up" by circles and of surfaces as "made up" of lines, he is not misusing the method of infinitesimals nor treading on dangerous ground. In fact the elements which "fill up" the magnitudes are always so taken by Archimedes that the process can be immediately satisfied by an exhaustion proof. From this point of view his scheme may be regarded as analogous to the modern method of infinitesimals when founded upon the doctrine of limits. There is a normal and systematic procedure, although tedious and laborious, for converting the mechanical proofs into exhaustion proofs. Therefore Archimedes may be regarded as having taken the decisive step in founding a method which in essential respects is that of the integral calculus. If he had been like many modern mathematicians, he would have omitted the exhaustion proofs altogether, but would have added to each of his mechanical proofs a set phrase like this: "It is easy to see that an exhaustion proof may be constructed in the usual manner. This is left as an exercise for the reader."

CHARLES S. SLICHTER.

SHORTER NOTICES.

Mehrdimensionale Geometrie, II Teil, *Die Polytope*. Von PROFESSOR DR. P. H. SCHOUTE. Leipzig, G. J. Göschen (Sammlung Schubert XXXVI). 1905. ix + 326 pp.

THE second (and final) volume of this work, like the first, is worthy to be associated with the other excellent books of the Schubert collection. Comparatively little of the subject matter is new, but a large number of interesting and useful results have been gathered together in a convenient form. The entire volume is devoted to the treatment of the polytop, which the author defines, for space of n dimensions, as any portion of that space enclosed in any manner whatever. The first 262 pages treat the linear polytop, *i. e.*, one bounded by flat spreads of $n-1$ dimensions (R_{n-1}'s); while the remaining 64 pages are concerned with the hypersphere, cone, cylinder, and rotation spread.

Under the heading "Topologische Einleitung," the first section treats (among other things) the simplex, and its various sections and projections; the question of a general classification of polytops; the definition of hyper-pyramids and prisms and the n-dimensional analogues of other special polyhedra, such as the truncated pyramid and prism, the frustums, etc.; and finally discusses the Euler law and its n-dimensional extension. In this treatment of the Euler law, thirteen pages are devoted to the well-known three-dimensional case, four different methods of proof being given, while the extension to n dimensions is disposed of in five pages. In view of the difficulty which the author mentions in the preface of reducing the volume to the required number of pages, it would seem that some space might have been saved to advantage at this point.

The second section is devoted to "Massverhältnisse." Congruence, both positive and negative, and similarity are defined, first for the simplex and then for the general linear polytop. Attention is called to three points metrically related to the simplex, viz., the in-center, the circumcenter, and the centroid or center of gravity. One regrets here that the author was forced to omit his originally intended paragraph on the geometry of the simplex. The orthocenter, so closely related to the three points mentioned, might well have been included. The important subject of the section is that of the content (Inhalt) of polytops. Formulas are deduced for the content of various types of polytops in the following order: The rectangular parallelotop, the general parallelotop, the prism, the simple pyramid, the simplex, etc., the general procedure being similar to the treatment in our elementary geometries. A convenient expression in determinant form for the content of a simplex in terms of the lengths of its $\binom{n+1}{2}$ edges is deduced.

"Reguläre Polytope" is the subject of the third section; and here again, in a work on higher-dimensional geometry, much less attention might have been given to the two- and three-dimensional cases. The larger part of the fifteen and forty-five pages devoted respectively to the regular polygon and regulae polyhedron might have been more advantageously used for some of the material which the author was forced to omit in condensing his manuscript. The treatment of the higher-dimensional polytops furnishes, in the estimation of the writer, the two most interesting chapters of the book. In one chapter the six four-dimensional figures are considered, the simplest sets of rectangular cartesian coordinates of the vertices are given for each, the Schlegel diagrams and various three-dimensional sections are studied, and the group connected with each figure is given. In the next chapter the three regular polytops which exist in space of any number of dimensions are studied. In a convenient little table one finds, for each of the three figures, formulas for the radii of the n associated hyper-spheres, the angle between two adjacent R_{n-1}'s, the content, and the boundary content (Oberfläche), all in terms of the length of an edge. The general expression for the order of the group associated with each of these figures is also given. The omission of a phrase from the theorem at the top of page 256 is easily corrected by reference to the formula at the bottom of the preceding page.

The fourth and final section treats "Die runden Polytope." In the work on the hyper-sphere one notices such topics as, the sphere passing through $n + 1$ points, the sphere touching $n + 1$ R_{n-1}'s, the sphere touching $n + 1$ other spheres, the configuration of the centers of similitude of $n + 1$ spheres, the content and surface content of the sphere, and the content of the spherical sector and segment. The hyper-cone and cylinder are similarly treated. Under general rotation figures, one finds the quadric spreads generated by revolving a flat spread about a flat spread as an axis, the torus spreads and the Guldin spreads obtained by revolving hyperspheres and linear polytops respectively.

Since the book is largely a compilation of previously known results, one regrets that references to the literature of the subject are not more numerous and specific. This second volume is better than the first in this respect, but still leaves much to be desired.

The book will doubtless prove to be a valuable reference work to those who are interested in, and have use for, the metrical formulas of higher-dimensional geometry; but many readers will doubtless share with the present writer a regret that the author's point of view has been so largely metrical, both in his choice of topics and in his method of treatment.

W. B. CARVER.

Theory of the Algebraic Functions of a Complex Variable. By J. C. FIELDS. Berlin, Mayer & Müller, 1906. v + 186 pp.

THE work before us is not intended as a treatise or textbook on the theory of algebraic functions along any of the well-established lines of treatment. It is, on the contrary, a new and distinctive mode of approach to this class of functions, although grounded on principles which in their essence are already familiar. The methods employed are purely algebraic, we might almost say arithmetic, in character, and in this respect the influence of Weierstrass may be said to predominate.

The fundamental idea on which the work is based is the notion of "order of coincidence." A given class of algebraic functions is defined as usual by a rational expression in (z, v)

the dependence of v on z being determined by an algebraic equation $F(v, z) = 0$ of degree n in v. The n branches of the function are assumed in the form $v = P_k(z - a)$ $(k = 1, 2, \cdots, n)$ in the vicinity of a given point $z = a$, the exponents in the power series being rational numbers only a finite number of which are negative. The order of coincidence of any function $H(z, v)$ with the branch $v - P_k = 0$ is then defined to be the least exponent obtained in developing H in powers of $z - a$, after substituting for v the series $P_k(z - a)$. For each point a there is accordingly a set of n (or fewer) numbers constituting the orders of coincidence of the given function with the several branches of the curve $F(v, z) = 0$. This set of numbers will be zero except for a finite number of points a. These sets of non-vanishing orders of coincidence are studied and the conditions which they must satisfy are determined.

On the other hand, if the orders of coincidence be completely given in advance, the function H is thereby conditioned, and it is then required to determine its form and properties. Chapter VI, for example, is devoted to finding the relation between the degree of a function and its orders of coincidence at infinity.

The methods of the author, growing as they do out of a single fundamental idea, are naturally characterized by a high degree of simplicity, unity, and generality. This is especially observable in Chapters XIII, XIV, and XV where the efficiency of the new treatment and its culminating "complementary theorem" of Chapter XII is tested by deducing for any algebraic relation in (z, v), reducible as well as irreducible, the Riemann-Roch and related theorems and formulas, and the general forms and properties of the ϕ-functions and abelian integrals. The brief discussion of the abelian integrals contained in the last chapter is merely intended as a suggestion of ways in which the previous theory might be utilized in the study of this class of transcendental functions.

As remarked in the preface, the methods of this work, at least in part and with suitable modifications, are applicable outside the field of algebraic functions and to functions of any number of variables.

The book is printed in a style uniform with the elegant edition of Weierstrass's works issued by the same publishers.

J. I. HUTCHINSON.

NOTES.

THE April number (volume 9, number 2) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Representations of the general symmetric group as linear groups in finite and infinite fields," by L. E. DICKSON; "Surfaces with isothermal representation of their lines of curvature and their transformations," by L. P. EISENHART; "The equilong transformations of space," by J. L. COOLIDGE; "Concerning linear substitutions of finite period with rational coefficients," by A. RANUM; "On hypercomplex number systems belonging to an arbitrary domain of rationality," by R. B. ALLEN; "On the asymptotic character of the solution of certain linear differential equations containing a parameter," by G. D. BIRKHOFF; "On the holomorph of the cyclic group of order p^m," by G. A. MILLER; "On non-measurable sets of points," by E. B. VAN VLECK.

THE April number (volume 30, number 2) of the *American Journal of Mathematics* contains: "Concerning systems of conics lying on cubic, quartic, and quintic surfaces," by C. H. SISAM; "On the canonical forms and automorphs of ternary cubic forms," by L. E. DICKSON; "The elliptic cylinder function of class K," by W. H. BUTTS; "On elliptic modular equations for transformations of orders 29, 31, 37," by A. BERRY; "On translation surfaces connected with a unicursal quartic," by J. EIESLAND.

AT the meeting of the London mathematical society held on March 12 the following papers were read: By E. B. ELLIOTT, "On the projective geometry of some covariants of a binary quintic"; by W. H. YOUNG, "On the inequalities connecting the double and repeated upper and lower integrals of a function of two variables"; by W. F. SHEPPARD, "On the operational expression of Taylor's theorem"; by H. A. P. DE S. PITTARD, "A proof of a theorem of Fermat's"; by M. J. M. HILL, "On a formula for the sum of a finite number of terms of the hypergeometric series, whose fourth element is unity."

THE summer meeting of the American association for the advancement of science will be held at Dartmouth College, Hanover, N. H., during the week beginning June 29.

THE tenth regular meeting of the association of teachers of mathematics in the Middle States and Maryland was held at the Woman's College, Baltimore, March 14. The following papers were read: By L. S. HULBERT, "Undergraduate instruction in mathematics"; by I. J. SCHWATT, "Our duty as teachers"; by W. H. JACKSON, "Notes on the teaching of mathematics in English preparatory schools and colleges"; by F. MORLEY, "A test of elementary text books in geometry"; by A. J. GMINDER, "The history of mathematical symbolism."

THE Manchester mathematical society was organized at Manchester, England, February 19, 1908, with an initial membership of 60. Professor H. LAMB was elected president, and Mr. H. BATEMAN secretary. Meetings will be held bimonthly. For the present the society will not undertake the publication of its proceedings.

THE following courses in mathematics are announced by German universities for the summer semester of 1908.

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: On space curves and curved surfaces, four hours; Calculus of variations, four hours; Applications of elliptic functions, two hours; Seminar, two hours; Colloquium, two hours.—By Professor G. FROBENIUS: Theory of determinants, four hours; Seminar, two hours.—By Professor F. SCHOTTKY: Elementary analysis, four hours; Theory of abelian functions, four hours; Seminar, two hours.—By Professor G. HETTNER: Introduction to the theory of ordinary differential equations, two hours.—By Professor J. KNOBLAUCH: Applications of elliptic functions, four hours; Calculus, one hour.—By Professor R. LEHMANN-FILHÈS: Analytic mechanics, four hours.—By Professor E. LANDAU: Differential calculus, four hours; Distribution of prime numbers, four hours.—By Dr. I. SCHUR: Theory of algebraic equations, II, four hours; Analytic geometry, four hours.

UNIVERSITY OF GÖTTINGEN.—By Professor F. KLEIN: Encyclopedia of geometry, four hours; Seminar, two hours.—By Professor D. HILBERT: Principles of mathematics, four hours; Seminar, two hours.—By Professor H. MINKOWSKI: Analytic geometry, four hours; Fourier series and definite integrals, two hours; Seminar, two hours.—By Professor C.

RUNGE: Differential and integral calculus with exercises, six hours; Graphical statics, one hour; Seminar, two hours.—By Professor L. PRANDTL: Theory of elasticity and rigidity, three hours; Introduction to the theory of machines, one hour; Seminar, two hours.—By Professor G. HERGLOTZ: Equilibrium and motion of liquids under gravity, three hours; Selected chapters of celestial mechanics, three hours.—By Professor E. ZERMELO: Mathematical logic, two hours.—By Dr. C. CARATHÉODORY: Functions of real variables, two hours; Historical introduction to the calculus of variations, one hour.—By Dr. P. KOEBE: Elementary differential equations, six hours.—By Dr. O. TOEPLITZ: Introduction to the theory of integral equations, three hours; Proseminar, two hours.—By Professor F. BERNSTEIN: Probabilities, two hours; History of mathematics, two hours.

UNIVERSITY OF LEIPZIG.—By Professor C. NEUMANN: Analytic mechanics, four hours.—By Professor O. HÖLDER: Applications of elliptic functions, three hours; Elliptic modular functions, three hours; Seminar, two hours.—By Professor F. HAUSDORFF: Differential equations, four hours.—By Professor K. ROHNS: Algebraic curves, four hours; Descriptive geometry, I, two hours.—By Professor H. LIEBMANN: Plane analytic geometry, four hours; Vector analysis with applications, two hours.

THE following advanced courses are announced by the various American universities for the academic year 1908–1909:

CORNELL UNIVERSITY.—By Professor J. MCMAHON: Theory of sound, two hours; Electricity, two hours.—By Professor J. I. HUTCHINSON: Theory of functions of a complex variable, three hours.—By Professor V. SNYDER: Higher geometry, three hours.—By Professor W. B. FITE: Theory of groups, three hours.—By Dr. F. R. SHARPE: Theory of potential and Fourier's series, three hours; Elliptic functions, two hours, first half year.—By Dr. W. B. CARVER: Projective geometry, three hours.—Dy Dr. A. RANUM: Elementary differential equations, two hours; Higher algebra, two hours.—By Dr. D. C. GILLESPIE: Advanced calculus, three hours; Integral equations, two hours, second half year.—By Dr. C. F. CRAIG: Advanced analytic geometry, three hours;

Partial differential equations, two hours, first
Dr. F. W. OWENS: Solid analytic geometry,
The Oliver mathematical club will meet we

INDIANA UNIVERSITY. — By Professor R.
vanced calculus, three hours (a, w, s); Hig
hours (a, w); Algebra of quantics, three hou
fessor S. C. DAVISSON: Ordinary differential
hours (a, w); Functions of a complex variabl
Fourier's series and integrals, three hours (a)
geometry, two hours (w, s). — By Professor D
Quaternions, three hours (a); Partial diffe
three hours (w, s). — By Professor U. S.
integrals and functions, two hours (a, w); I
products, three hours (s). — By Dr. C. H.
matical theory of elasticity, three hours (a, w
tential, three hours (s). [a, w, s above indi
ter, and spring terms.]

YALE UNIVERSITY. — By Professor J. P
duction to the theory of functions, two
geometry, two hours; Advanced mechanics,
vanced theory of functions, two hours. — By
SMITH: Advanced analytic geometry, two h
groups of transformations, two hours. — By
BROWN: Mechanics, two hours; Advance
hours; Celestial mechanics, two hours. — By
HAWKES: Algebra and analytic geometry, t
of equations, two hours. — By Professor M
differential equations, two hours; Calculus
hour. — By Dr. L. I. HEWES: Differenti
hour; Graphical and numerical computatio
Dr. W. A. GRANVILLE: Differential geome

THE following advanced summer courses a

UNIVERSITY OF CHICAGO (summer qua
August 28, 1908).— By Professor H. S. W
elliptic functions, four hours; Higher plane
— By Professor H. E. SLAUGHT: Theory of
four hours; Analytic geometry, five hours. -
W. A. YOUNG: Differential calculus, five
of secondary mathematics, four hours. — B

DICKSON: Theory of equations, four hours. — By Professor G. A. BLISS: Differential geometry from the Lie standpoint, four hours; Integral calculus, five hours. — By Dr. A. C. LUNN: Analytic mechanics, four hours; Higher algebra, five hours.

COLUMBIA UNIVERSITY (summer session, July 7 to August 14). — By Professor JAMES MACLAY: Advanced calculus, five hours. — By Professor W. B. SMITH: Theory of functions of a complex variable, five hours. — By Professor EDWARD KASNER: Differential equations, five hours. — By Dr. G. H. LING: Modern higher algebra, five hours.

INDIANA UNIVERSITY (summer session, June 25 to Sept. 4). — By Professor R. J. ALEY: History of mathematics, three hours; Ordinary differential equations, four hours. — By Professor S. C. DAVISSON: Advanced integral calculus, five hours; Theory of surfaces, five hours. — By Professor D. A. ROTHROCK: Solid analytic geometry, three hours.

UNIVERSITY OF PENNSYLVANIA (summer session, July 6 to August 15). — By Professor I. J. SCHWATT: Definite integrals, five hours. — By Professor G. H. HALLETT: Theory of functions of a complex variable, five hours. — By Professor F. H. SAFFORD: Differential equations, five hours.

THE following Smith's prizemen are announced for the year 1908: W. J. HARRISON (Clare College) for his essay, "Problems in the wave motion of viscous liquids"; J. E. LITTLEWOOD (Trinity College) for his essay, "On the asymptotic behavior of integral functions of zero order, and allied functions"; J. MERCER (Trinity College) for his essay, "On the solution of ordinary linear differential equations having doubly periodic coefficients." Five other essays were declared worthy of honorable mention.

THE Hon. BERTRAND RUSSELL has been elected a fellow of the Royal society of London.

PROFESSOR P. STÄCKEL, of the technical school at Hanover, has accepted a call as professor of mathematics at the technical school at Karlsruhe.

PROFESSOR V. VOLTERRA, of the University of Rome, has been elected foreign member of the royal academy of sciences of Stockholm.

PROFESSOR H. POINCARÉ, of the University of Paris, has been elected to membership in the French academy, as successor to the late M. SULLY PRUDHOMME. Professor Poincaré has been a member of the academy of sciences since 1887.

THE Institute of France has awarded the Binoux prize (for the history of the sciences) for the current academic year to Professor GINO LORIA, of the University of Genoa.

DR. E. HILB has been appointed docent in mathematics at the University of Erlangen.

AT the University of Nebraska, Professor C. C. ENGBERG has been promoted to a full professorship of applied mathematics, and Professor W. C. Brenke has been promoted to an assistant professorship of mathematics.

MR. B. F. MORRELL has been appointed professor of mathematics at Fort Worth University, Fort Worth, Texas.

AT Harvard University, Dr. J. L. Coolidge has been promoted to an assistant professorship of mathematics.

MR. P. B. TURNER has been appointed tutor in mathematics at the College of the City of New York.

PROFESSOR C. H. ASHTON, of the University of Kansas, has been granted leave of absence for the next academic year, and will study in Munich.

PROFESSOR L. LINDELÖF, of the University of Helsingfors, died March 3, at the age of 70 years.

DR. M. P. H. LAURENT, examiner in mathematics at the Ecole polytechnique of Paris, died March 2, at the age of 67 years.

PROFESSOR L. WEDEKIND, of the technical school at Karlsruhe, died February 8, at the age of 69 years.

PROFESSOR A. E. v. BRAUNMÜHL, of the technical school at Munich, died March 11, at the age of 55 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Björnbo (A. A.). See Vernerus (I.).

Camacho y Sanjurjo (J. J.). Invariantes; elementos de geometría analítica. Sevilla, Diaz, 1907. 23 pp.

Dingler (H.). Beiträge zur Kenntnis der infinitesimalen Deformationen einer Fläche. (Diss.) München, 1907. 8vo. 62 pp.

Dressler (H.). Die Lehre von der Funktion. Theorie und Aufgabensammlung für alle höheren Lehranstalten. (Mittelschulen.) Leipzig, Dürr, 1908. 8vo. 93 pp. M. 1.60

Fulco (P.). I coefficienti delle equazioni differenziali lineari omogenee, ammettenti fra i loro integrali particolari funzioni $\alpha(x)$. Memoria. Lodi, Wilmant, 1908. 8vo. 10 pp.

Geigenmüller (R.). Leitfaden und Aufgabensammlung zur höheren Mathematik. Für technische Lehranstalten und den Selbstunterricht bearbeitet. Vol. II. Die höhere Analysis oder Differential- und Integralrechnung. 6te Auflage. Mittweida, 1908. 8vo. 8 + 339 pp. Cloth. M. 7.00

Heger (R.). Analytische Geometrie auf der Kugel. (Sammlung Schubert, LIV.) Leipzig, Göschen, 1908. 12mo. 7 + 152 pp. Cloth. M. 4.40

Haseman (C.). Anwendung der Theorie der Integralgleichungen auf einige Randwertaufgaben in der Funktionentheorie. (Diss.) Göttingen, 1907. 8vo. 46 pp.

Junker (F.). Repetitorium und Aufgabensammlung zur Integralrechnung. Leipzig, 1907. 12mo.

Keyser (C. J.). Mathematics. (A lecture delivered at Columbia University.) New York, Columbia University Press, 1907. 8vo. 44 pp. $0.25

Köstlin (E.). Ueber eine Deutung der Gleichung, die zwischen dem Bogen und dem Neigungswinkel der Tangente im Endpunkt des Bogens einer ebenen Kurve besteht. (Diss.) Tübingen, 1907. 8vo. 60 pp.

Löffler (E.). Beiträge zur Theorie der Schnittpunkte algebraischer Kurven. (Diss.) Tübingen, 1907. 8vo. 49 pp.

Ratschläge und Erläuterungen für die Studierenden der Mathematik und Physik an der Universität Göttingen. Herausgegeben von der Direktion des mathematisch-physikalischen Seminars. Neue Auflage. Mit einem Anhang: Statuten des mathematischen Lesezimmers. Leipzig, Teubner, 1907. 8vo. 31 pp.

Reineck (A.). Die Verwandtschaft zwischen Kugelfunktionen und Besselschen Funktionen. (Diss., Bern.) Halle a. S., 1907. 8vo. 72 pp.

Ronco (N.). Studio geometrico delle reciprocità; tesi per la laurea in matematica pura. Torino, Streglio, 1907. 8vo. 45 pp.

Schoenflies (A.). Die Entwickelung der Lehre von den Punktmannigfaltigkeiten. Bericht, erstattet der deutschen Mathematiker-Vereinigung. 2er Teil. (Jahresbericht der deutschen Mathematiker-Vereinigung, Ergänzungsband II.) Leipzig, Teubner, 1908. 8vo. 10 + 331 pp.

Schübel (H.). Aufstellung von nicht-euklidischen Minimalflächen. (Diss.) München, 1906. 8vo. 47 pp.

Vernerus (J.). Ioannis Verneri de triangulis sphæricis libri quatuor, de meteoroscopiis libri sex, cum prooemio G. I. Rhetici. I. De triangulis sphæricis. Herausgegeben von A. A. Björnbo. (Abhandlungen zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, begründet von M. Cantor, 24es Heft, 1te Abteilung.) Leipzig, Teubner, 1907. 8vo. 12 + 184 pp.

II. ELEMENTARY MATHEMATICS.

Bardey (E.). Arithmetische Aufgaben nebst Lehrbuch der Arithmetik. 2er Teil. (Für die Oberklassen neunstufiger Anstalten.) Bearbeitet von H. Hartenstein. Leipzig, Teubner, 1907. 8vo. 5 + 230 pp. M. 2.60

Behacker. See Močnik.

Bos (H.). Géométrie élémentaire. 21e édition. Paris, Hachette, 1908. 16mo. 288 pp. Fr. 2.00

Donadt (A.). See Lübsen (H. B.).

Dupuis (J.). Tables de logarithmes à cinq décimales contenant les logarithmes des nombres entiers de 1 à 10000, les logarithmes des sinus et des tangentes calculés de minute en minute jusqu'à 90 degrés, plusieurs tables usuelles et un grand nombre de formules et de nombres utiles. 29e édition. Paris, Hachette, 1908. 16mo. 4 + 231 pp. Fr. 2.00

Fuss (K.). Resultate, Lösungs-Andeutungen und ausführliche Auflösungen zu den Aufgaben aus der Buchstabenrechnung und Algebra. Für Schulen und zum Selbstunterricht. 6te, vermehrte und verbesserte Auflage. Nürnberg, Korn, 1908. 8vo. 8 + 325 pp. M. 3.20

Grévy (A.). Géométrie théorique et pratique à l'usage de l'enseignement secondaire des jeunes filles (classes de 4e et 5e années). Paris, Vuibert, 1908. 8vo. 197 pp.

Hartenstein (H.). See Bardey (E.).

Hočevar (F.). Lehr- und Uebungsbuch der Geometrie für Untergymnasien. 8te Auflage. Wien, Tempsky, 1907. 8vo. 123 pp. M. 1.80

Jacquemart (A.). Algèbre pratique en quinze leçons, à l'usage des écoles primaires supérieures et des écoles normales. 5e édition. Paris, Larousse, 1908. 16mo. 112 pp. Fr. 1.00

Krünes (F.). See Močnik.

Lübsen (H. B.). Ausführliches Lehrbuch der ebenen und sphärischen Trigonometrie. Zum Selbstunterricht mit Rücksicht auf die Zwecke des praktischen Lebens bearbeitet. Neu bearbeitet von A. Donadt. 19te Auflage. Leipzig, Brandstetter, 1908. 8vo. 7 + 146 pp. M. 2.90

Matriculation mathematics papers. From Jan. 1897 to Jan. 1908. New edition. London, Clive, 1908. 8vo. 138 pp. 1s. 6d.

—— model answers: mathematics. From Sept. 1904 to Jan. 1908. New edition. London, Clive, 1908. 8vo. 161 pp. 2s.

Močnik. Lehrbuch der Arithmetik für Untergymnasien. Bearbeitet von A. Neumann. 1te Abteilung für die erste und zweite Klasse. 39te, im wesentlichen unveränderte Auflage. Wien, Tempsky, 1907. 8vo. 3 + 148 pp. M. 2.30

Močnik und Behacker. Lehrbuch der Arithmetik für Lehrer- und Lehrerinnen-Bildungsanstalten. Bearbeitet von F. Krünes. 8te, durchgesehene und berichtigte Auflage. Wien, Tempsky, 1907. 8vo. 262 pp. M. 2.90

Papelier (G.). Formulaire de mathématiques spéciales: algèbre, analyse, trigonométrie, géométrie analytique. 2e édition, revue et rendue conforme au programme de la classe de mathématiques spéciales. Paris, Vuibert, 1908. 8vo. 212 pp.

——. Précis d'algèbre, d'analyse et de trigonométrie, à l'usage des élèves de mathématiques spéciales. 2e édition, mise en harmonie avec le nouveau programme. Paris, Vuibert, 1908. 8vo. 452 pp.

Romero (A.). Nociones y ejercicios de geometría. Alicante, Rovira, 1907. 94 pp.

Schindler (A.). Leitfaden für den Unterricht in der praktischen Geometrie an der k. k. technischen Militär-Akademie. Im Auftrage des k. k. Reichskriegsministeriums verfasst. 2er Teil: Feldmesskunst; Höhenmesskunst; besondere Messverfahren; Anhang. Wien, Seidel, 1907. 8vo. 5 + 153 pp. Cloth. M. 7.50

Schultze (A.). Graphic algebra. New York, Macmillan, 1908. 12mo. 8 + 93 pp. Cloth. $0.80

Somerville (F. H.). Elementary algebra. New York, American Book Co., 1908. 12mo. 3 + 407 pp. Cloth. $1.00

——. Answers to Elementary algebra. New York, American Book Co., 1908. 12mo. 26 pp. $0.10

Spinal (G.). Elementi d'algebra, ad uso delle scuole tecniche. Catania, Battiato, 1907. 8vo. 128 pp. L. 1.65

Wheeler (A. H.). Algebra for grammar schools, with mental exercises. Boston, Little, Brown and Co., 1907. 12 mo. 12 + 186 pp. Cloth. $0.50

——. First course in algebra; with 8000 examples, including 3000 mental exercises. Boston, Little, Brown and Co., 1907. 12 mo. 19 + 664 pp. Cloth. $1.15

III. APPLIED MATHEMATICS.

Barchanek (K.). Lehr- und Uebungsbuch der darstellenden Geometrie für Oberrealschulen. 2te, im Sinne des Normallehrplanes gekürzte Auflage. Wien, Tempsky, 1907. 8vo. 208 pp. M. 3.20

Berniolle (P.). Leçons de géométrie descriptive conformes aux programmes de 1905, pour les classes de première *C* et *D*, de mathématiques *A* et *B*, préparation aux Ecoles navales et de Saint-Cyr et à l'Institut agronomique. 2e partie. 2e édition, entièrement refondue. Paris, Paulin, 1908. 18mo. 191 pp. Fr. 4.00

Caronnet (J.). Problèmes de mécanique, à l'usage des élèves des classes de mathématiques *A* et *B* et des candidats au baccalauréat et aux écoles du gouvernement. 2e édition, mise en harmonie avec les nouveaux programmes. Paris, Vuibert, 1908. 8vo. 328 pp.

Dietz (E.). Ueber die Anwendung der darstellenden der astronomischen Geographie. (Diss.) Basel

French (L. G.). First principles of theoretical m Industrial Press, 1908. 8vo. 37 pp.

Fuchs (F.). Beiträge zur Theorie der elektrische leitenden Rotationsellipsoides. (Diss.) Münch

Haber (F.). Thermodynamics of technical gas-react Translated by A. B. Lamb. New York, L 19 + 356 pp. Cloth.

Lamb (A. B.). See Haber (F.).

Lewe (V.). Die plötzlichen Fixierungen eines Beitrag zur vektoranalytischen Behandlung der tankräfte. (Diss.) Tübingen, 1906. 8vo. 25

Pelletan (A.). Programme du cours de géomét Paris, Béranger, 1907. 8vo. 8 pp.

Petit (E.). See Richard (P. J.).

Richard (P. J.) et Petit (E.). Théorie mathém Paris, Doin, 1908. 18mo. 400 + 12 pp.

Schleschka (J.). Lehrbuch der Mechanik für de meisterschulen und gleichartig organisierten tech Wien, Pichler, 1907. 8vo. 7 + 304 pp.

Turner (G. C.). Graphics; applied to arithmetic, n New York, Macmillan, 1908. 12mo. 9 + 388 p

Weinstein (B.). Thermodynamik und Kinetik de 2er Halbband. Thermodynamik der Elektrizität (2er Teil); Elektrochemie. Braunschweig, 20 pp. + pp. 465–1186.

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THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and thirty-eighth regular meeting of the Society was held in New York City on Saturday, April 25, 1908, extending through the usual morning and afternoon sessions. The attendance included the following thirty-five members:

Professor G. A. Bliss, Professor Maxime Bôcher, Professor F. N. Cole, Professor L. P. Eisenhart, Professor G. H. Hallett, Mr. G. W. Hartwell, Dr. A. A. Himowich, Professor E. V. Huntington, Professor J. I. Hutchinson, Professor Edward Kasner, Professor C. J. Keyser, Mr. E. H. Koch, Jr., Mr. W. C. Krathwohl, Professor P. A. Lambert, Dr. G. H. Ling, Mr. L. L. Locke, Mr. E. B. Lytle, Dr. Emory McClintock, Professor Max Mason, Mr. A. R. Maxson, Dr. R. L. Moore, Professor Richard Morris, Professor W. F. Osgood, Mr. H. W. Reddick, Mr. E. W. Sheldon, Mr. L. P. Siceloff, Professor S. E. Slocum, Mr. F. H. Smith, Professor P. F. Smith, Dr. C. E. Stromquist, Professor H. D. Thompson, Mr. C. A. Toussaint, Professor Oswald Veblen, Professor H. S. White, Professor J. W. Young.

The President of the Society, Professor H. S. White, occupied the chair, being relieved at the afternoon session by Professor C. J. Keyser. The Council announced the election of the following persons to membership in the Society: Professor H. E. Buchanan, Lincoln College, Lincoln, Ill.; Mr. E. F. A. Carey, University of California; Professor F. E. Chapman, Southern University, Greensboro, Ala.; Professor R. C. Maclaurin, Columbia University; Mr. E. J. Miles, University of Chicago; Mr. C. A. Stiles, University Preparatory School, Ithaca, N. Y.; Mr. J. S. Thompson, Mutual Life Insurance Company, New York, N. Y.; Mr. O. A. Turney, Phoenix, Ariz.; Mr. C. B. Walsh, Ethical Culture School, New York, N. Y.; Professor R. T. Wilbur, Christian Brothers College, St. Louis, Mo.; Miss E. R. Worthington, Yale University. Ten applications for admission to membership in the Society were received.

At the meeting of the Council Professor E. B. Van Vleck was reelected a member of the Editorial Committee of the *Transactions*, to serve for the three years 1908–1911. It was decided to hold the summer meeting and colloquium of the Society in 1909 at Princeton University, and Professors Fine, Osgood, Holgate, and the Secretary were appointed a committee to make the appropriate arrangements. A committee consisting of Professor P. F. Smith, C. J. Keyser, and G. A. Bliss was appointed to consider the question of holding the next annual meeting of the Society at Baltimore in affiliation with the American association for the advancement of science.

The following papers were read at this meeting:

(1) Professor S. E. Slocum: "The collapse of tubes under external pressure."

(2) Professor E. B. Wilson: "On the differential equations of the equilibrium of an inextensible string."

(3) Professor E. B. Wilson: "On the principle of relativity."

(4) Dr. Elijah Swift: "Note on the second variation in an isoperimetric problem."

(5) Professor J. I. Hutchinson: "The hypergeometric functions of n variables."

(6) Professor Edward Kasner: "Note on Meusnier's theorem."

(7) Professor B. F. Finkel: "Determination of the groups of order 2^m which contain selfconjugate cyclic subgroups of order 2^{m-4} and whose generating operations correspond to the partitions $[m-4, 4]$, $[m-4, 3, 1]$."

(8) Professor J. W. Young: "Two-dimensional chains and the classification of complex collineations in a plane."

(9) Mr. C. N. Moore: "On certain constants analogous to Fourier's constants."

(10) Professor Paul Saurel: "On the distance from a point to a surface."

(11) Mr. E. B. Lytle: "Multiple integrals over iterable fields."

(12) Professor P. A. Lambert: "The fundamental theorem of algebra."

(13) Professor E. V. Huntington and Mr. H. P. Forté: "On the fluctuations in the speed of a flywheel."

(14) Professor E. V. Huntington: "On the theory of the gyroscope, with special reference to the Brennan monorail car" (preliminary communication).

(15) Dr. O. E. Glenn: "Studies in the theory of degenerate algebraic curves" (preliminary communication).

In the absence of the authors the papers of Professor Wilson, Dr. Swift, Professor Finkel, Mr. Moore, Professor Saurel, and Dr. Glenn were read by title. The papers of Dr. Swift and Mr. Moore appeared in the May BULLETIN. Professor Saurel's paper will be published in the July number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Slocum's paper is an analysis of experimental data on the collapse of tubes under external pressure. The material for this analysis is found in the recent extensive experiments made independently by Professor Carman and Professor Stewart.

The rational formulas for thin and thick tubes are discussed, and the significance of the elastic constants involved, as indicating the nature of collapse, is pointed out. The fact that the physical properties of the material, given in connection with the experiments in question, did not include these elastic constants, indicates the irrational nature of the results and their consequent limitations. The characteristic difference between the failure of thin and thick tubes is also explained.

The nature and amount of the deviation of commercial tubes from the ideal assumptions upon which the rational formulas are based is examined and corrections determined. In this connection the flexibility of rational formulas as applying to particular cases, and the rigidity of empirical formulas as merely striking an average, are contrasted in terms of numerical percentages.

A graphical comparison of the curves represented by the rational formulas for thin and thick tubes is made for steel and for brass. The limit for use is determined as the abscissa of the point of intersection of these curves; and is found to agree closely with the empirical values determined experimentally.

A graphical comparison is also made between Stewart's empirical formulas and the corrected rational formulas for thin and thick tubes, the former representing an ellipse and its linear tangent, and the latter a cubic curve and a parabola respectively.

The results of the analysis are embodied in the formulas

$$P = C\frac{2E}{1-m^2}\left(\frac{t}{D}\right)^3 \text{ for thin tubes,}$$

$$P = 2u_c K\frac{t}{D}\left(1 - K\frac{t}{D}\right) \text{ for thick tubes,}$$

where $E =$ Young's modulus, $m =$ Poisson's ratio, $u_c =$ ultimate compressive strength, $P =$ external collapsing pressure, $t =$ thickness, $D =$ diameter, and C and $K =$ correction constants, all dimensions being expressed in inches and pounds as units. C and K are so expressed that they may be determined independently for individual specimens, or for the various types of commercial tubes. From the data under discussion C and K are found to have the following values: For lap welded steel boiler flues, $C = .69$ and $K = .89$; for cold drawn seamless steel flues, $C = .76$; for drawn seamless brass tubes, $C = .78$. The article will appear in full in the *Engineering News*.

2. Professor Wilson treats the differential equations of the equilibrium of an inextensible string from the point of view of Lie's theory of groups. He obtains the conditions that the equations should admit a group of eight parameters, and thereby determines a case in which the equations are completely integrable by quadratures and where the field of force is neither central nor parallel. The paper will be offered to the *Transactions*.

3. In a short note, Professor Wilson discusses the principle of relativity in electrodynamics with reference to recent suggestions and recent experimental developments. The paper will be offered to the *Philosophical Magazine*.

5. The $n + 1$ linearly independent hypergeometric integrals regarded as functions of n parameters x_i entering into the integrand are considered in the paper by Professor Hutchinson. The chief problem solved is the determination of the group of homogeneous linear transformations which these integrals undergo when the x_i describe closed paths. It is also shown that a certain hermitian form is invariant for this group.

6. Professor Kasner shows that the most general system of space curves for which Meusnier's theorem is valid is defined by a differential equation of the type $\alpha y'' + \beta z'' + \gamma = 0$, where the coefficients are arbitrary functions of x, y, z, y', z'. This includes Lie's extension as a special case. Such systems arise in connection with the general transformation of lineal elements.

8. Fundamental in the classification of the linear fractional transformations on one complex variable — *i. e.*, of the projectivities on a complex line — is the notion of a chain of points on the line. A chain may be defined as any class of points projective with the real points of the line. If the complex numbers be represented in the usual manner by the real points of a plane, the chains of the complex line are represented by the circles of this plane. The behavior of the projectivities on the line with reference to the chains of the line then leads readily to the well-known classification of these projectivities into hyperbolic, elliptic, etc.*

The corresponding classification of the projective transformations in a complex space of n dimensions must lead first to a generalization of the notion of a chain. It is obtained naturally as follows: Using homogeneous coordinates, any point in a complex space S_n of n dimensions is represented by $(x_0, x_1, x_2, \cdots, x_n)$, where the x_i are any complex numbers (the combination $(0, 0, \cdots, 0)$ of course excluded). The totality of such points in which the x_i are any real numbers forms the real subspace R_n of n dimensions in S_n. Any subspace of S_n obtained by subjecting R_n to any linear homogeneous transformation on the x_i with complex coefficients Professor Young calls an n-dimensional chain in S_n, or more briefly an n-chain.† In the present paper Professor Young considers in detail the case $n = 2$. Certain fundamental properties of 2-chains in a complex plane are first developed, which are then applied to effect the desired classification "with respect to reality" of the complex collineations in a plane. Each of the well-known five types of collineations in the plane is subdivided with reference to the existence of invariant 2-chains, and the existence of each subtype is established. This classification is of interest not merely in the field of projective geometry and continuous groups, but also in the theory of automorphic functions of two variables. The treatment is largely synthetic and the proofs are so simple as to make the methods applicable to larger values of n with comparatively little increase in complexity.

11. Former discussions of the problem of the reduction of

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* For a synthetic treatment of the one-dimensional case, cf. a paper by the author presented to the Society at its last (February) meeting on "The geometry of chains on a complex line."

† For a non-analytic definition of this notion, cf. "A system of assumptions for projective geometry," by O. Veblen and J. W. Young, presented to the Society at its last Christmas meeting.

multiple integrals to iterated integrals have been restricted to functions defined over Jordan measurable fields. Mr. Lytle defines iterable fields in such a way as to include all measurable and a large class of non-measurable aggregates. By means of this new class of iterable aggregates important relations between multiple and iterated integrals are derived. These relations lead to a very general condition for inversion of iterated integrals.

12. The purpose of Professor Lambert's paper is to present a general method for determining all the roots of any algebraic equation by means of infinite series. The method consists in forming three algebraic functions of x from the given equation (1) $f(y) = 0$, (a) by introducing a factor x into all the terms of (1) except the first and last; (b) by introducing a factor x into all the terms of (1) except the first and second; (c) by introducing a factor x into all the terms of (1) except the second and last.

These algebraic functions are expanded into power series in x by Laplace's series. If in these power series x is made unity, the resulting series, if convergent, determine the roots of the given equation. It is shown that all the roots of the algebraic equation can be expressed in infinite series derived either from the algebraic function formed in accordance with (a), or from the two algebraic functions formed in accordance with (b) and (c).

13. The paper of Professor Huntington and Mr. Forté contains an exact formula for the small fluctuations in the speed of the fly wheel of a horizontal engine, with the results of illustrative experiments made in the Harvard engineering laboratory by the use of a tachograph.

Let N be the number of revolutions per minute, regarded as a function of the crank angle θ, and suppose that $N = N_1$ when $\theta = \theta_1$. Then the value of N for any other value of θ is given by

$$N^2 = N_1^2 \left\{ 1 + \frac{\dfrac{1800g}{\pi^2 N_1^2 r}\left(\dfrac{U}{W_p r}\right)\Bigg|_{\theta=\theta_1}^{\theta=\theta} - \left(A^2 + \dfrac{w}{W_p} B^2\right)\Bigg|_{\theta=\theta_1}^{\theta=\theta}}{\dfrac{W}{W_p}\dfrac{K^2}{r^2} + A^2 + \dfrac{w}{W_p} B^2} \right\},$$

where A and B are numerical functions of θ depending on the proportions of the engine, and U is the total work done by the external forces (steam pressure, load, gravity, and friction) during the interval from $\theta = 0$ to $\theta = \theta$. Here W, W_p, w are the weights of flywheel, piston, and connecting rod, K = radius of gyration of the flywheel, and r = crank radius. The values of A and B are as follows:

$$A = \frac{\sin(\theta + \phi)}{\cos\phi}, \text{ where } \phi \text{ is given by } \sin\phi = \frac{r}{l}\sin\theta;$$

$$B^2 = \sin^2\theta\left[1 + \frac{l-c}{l}\frac{r}{l}\frac{\cos\theta}{\cos\phi}\right]^2 + \frac{c^2}{l^2}\cos^2\theta + \frac{k^2}{r^2}\left(\frac{r}{l}\frac{\cos\theta}{\cos\phi}\right)^2.$$

Here l = length of connecting rod, c = distance from wrist pin to center of gravity, and k = radius of gyration about the center of gravity. For the value of U, if the effective pressure P on the piston and the resisting moment RL of the load L about the shaft are known functions of θ, and if the expression for the work of friction is assumed to be of the form $(F + fL)R$, where F and f are constants, then

$$U = \int_0^\theta [PdX - RLd\theta - (F + fL)Rd\theta] - wc\sin\phi,$$

where $X = r(1 - \cos\theta) + l(1 - \cos\phi)$, so that $dX = rAd\theta$.

The formula for N holds good for all conditions of pressure and load; the condition for "uniform speed" is that the integral in the expression for U shall reduce to zero at the end of each revolution.

14. Professor Huntington's second paper, when completed, will contain approximate solutions of the equations of motion which present themselves in the theory of the gyrostatic balancing of the Brennan monorail car.

15. The general subject of the relations which exist between an algebraic curve and those curves of lower degree into which it can decompose has been studied only incidentally, or at best only in connection with special curves of low degrees. Certain results have been published by Cayley, Zeuthen, Taylor, and others; but their work is only fragmentary. Dr. Glenn has organized a general theory of the decomposition of curves of nth degree, and presents, in the form of a preliminary report approximately half of the results of his investigation.

The report contains chapters on the following topics: the asymptotic properties of ternary homogenous forms; the penultimate forms of a curve; the existence of "free summits" on a degenerate algebraic curve; the classification of curves on the basis of the penultimate forms. Further investigations are in progress. The memoir, as a whole, will be offered for publication the coming year.

F. N. COLE,
Secretary.

THE APRIL MEETING OF THE CHICAGO SECTION.

THE twenty-third regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, on Friday and Saturday, April 17–18, 1908. One session was held on Friday and two on Saturday. On Friday evening twenty of the members dined together informally in the private dining room of the University of Chicago Commons. The attendance at the three sessions included forty-five persons, among whom were the following thirty-three members:

Dr. G. D. Birkhoff, Professor Oscar Bolza, Dr. R. L. Börger, Professor H. E. Buchanan, Mr. Thomas Buck, Professor W. H. Bussey, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Mr. Arnold Dresden, Mr. E. P. R. Duval, Professor W. B. Ford, Mr. R. M. Ginnings, Professor Harriet E. Glazier, Professor C. N. Haskins, Mr. T. H. Hildebrandt, Mr. F. H. Hodge, Dr. A. C. Lunn, Mr. W. D. MacMillan, Professor G. A. Miller, Dr. J. C. Morehead, Professor F. R. Moulton, Dr. L. I. Neikirk, Professor H. L. Rietz, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor H. E. Slaught, Dr. Clara E. Smith, Dr. A. L. Underhill, Professor E. B. Van Vleck, Dr. A. E. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

Professor G. A. Miller, Vice-President of the Society and chairman of the Section, presided at all of the sessions. In opening the meeting he spoke of the great loss sustained by the Society in the recent death of Professor Heinrich Maschke. A committee, consisting of Professors E. B. Van Vleck, Alexander Ziwet, and H. E. Slaught, was appointed to draft suitable resolutions on behalf of the Section, and these were presented at the close of the session on Saturday morning, adopted by a rising vote, and ordered spread upon the minutes. The resolutions are printed immediately following this report. A memorial of Professor Maschke, prepared by Professor Bolza, will soon appear in the BULLETIN.

The following papers were read:

(1) Dr. C. H. SISAM: "On a locus determined by concurrent tangents."

(2) Professor W. B. FORD: "On the integration of the equation $a_0(x)\, u(x + 2) + a_1(x)\, u(x + 1) + a_2(x)\, u(x) = 0$."

(3) Professor D. R. CURTISS: "On the real branches of implicit functions in the neighborhood of multiple points."

(4) Mr. L. L. DINES: "A method of investigating numbers of the forms $6^\kappa \cdot s \pm 1$."

(5) Professor L. E. DICKSON: "Criteria for the irreducibility of a reciprocal equation."

(6) Professor L. E. DICKSON: "On reciprocal abelian equations."

(7) Professor L. E. DICKSON: "On the congruence $x^n + y^n + z^n \equiv 0 \pmod{p}$."

(8) Professor JACOB WESTLUND: "Note on the equation $x^n + y^n = nz^n$."

(9) Mr. F. H. HODGE and Mr. E. J. MOULTON: "On certain characteristics of orbits for a general central force."

(10) Professor G. A. MILLER: "The central of a group."

(11) Dr. A. E. YOUNG: "On the problem of the spherical representation and the characteristic equations of certain classes of surfaces."

(12) Dr. A. C. LUNN: "A continuous group related to von Seidel's optical theory."

(13) Dr. A. C. LUNN: "A minimal property of simple harmonic motion."

(14) Dr. A. C. LUNN: "The deduction of the electrostatic equations by the calculus of variations."

(15) Mr. A. R. SCHWEITZER: "Remark on Enriques's review of the foundations of geometry."

(16) Mr. A. R. SCHWEITZER: "On the calculi of relations, classes, and operations."

(17) Professor E. J. WILCZYNSKI: "Projective differential geometry of curved surfaces, fourth memoir."

(18) Dr. G. D. BIRKHOFF: "Irregular integrals of ordinary linear differential equations."

(19) Professor R. D. CARMICHAEL: "On the general tangent to plane curves."

(20) Professor R. D. CARMICHAEL: "On plane algebraic curves symmetrical with respect to each of two rectangular axes."

(21) Professor O. D. KELLOGG: "Note on the geometry of continuously turning curves."

(22) Dr. I. SCHUR: "Beiträge zur Theorie der Gruppen linearer homogener Substitutionen."

(23) Mr. W. D. MACMILLAN: "On the character of the solutions of homogeneous linear equations with periodic coefficients."

(24) Mr. A. R. SCHWEITZER: "On the quaternion as an operator in Grassmann's extensive algebra."

Mr. Dines was introduced by Professor Curtiss. Dr. Schur's paper was communicated to the Society through Professor Dickson. In the absence of the authors the papers of Dr. Sisam, Professor Westlund, Professor Wilczynski, Professor Carmichael, and Dr. Schur were read by title. Professor Dickson's first paper appears in the present number of the BULLETIN. His second paper has been published in the April number of the *American Mathematical Monthly*. Abstracts of the other papers follow below, the numbering being the same as that attached to the titles in the above list.

1. In the plane of an arbitrary algebraic curve C there exists a second curve C' such that the points of tangency of three or the tangents to C from each point of C' are collinear. Dr. Sisam's paper is devoted to the discussion of the locus C' and of the dual locus.

2. Professor Ford's paper considers the behavior for large values of x of the general solution of the equation

$$a_0(x)u(x+2)+a_1(x)u(x+1)+a_2(x)u(x)=0,$$

the coefficients $a_0(x)$, $a_1(x)$, $a_2(x)$ being given functions (real or complex) of x defined for all positive integral values of x sufficiently large. Especial attention is given to the important case in which the coefficients are developable in Maclaurin series about the point $x=\infty$. The results obtained may be employed in the study of the convergence of algebraic continued fractions, as will appear subsequently. The paper has been offered for publication in the *Transactions*.

3. In the paper of Professor Curtiss conditions for the existence of real branches of an implicit function $f(x, y) = 0$, in the neighborhood of a multiple point, are obtained by methods which are analogous to those used by Stolz and others in the theory of maxima and minima, and which constitute a generalization of the method of Dini for an ordinary point. The invariant character of these conditions is discussed, and the connection of semi-definite cases with the contact of branches of the curves

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

is indicated.

4. Mr. Dines's paper describes a method of determining factors of numbers of the forms $6^\kappa \cdot s \pm 1$ or primes of those forms, easily applicable to numbers above the limits of the existing factor tables. The x_κ and x'_κ functions of any prime p are defined respectively as the least positive solutions of the congruences $6^\kappa \cdot s - 1 \equiv 0 \pmod{p}$ and $6^\kappa \cdot s + 1 \equiv 0 \pmod{p}$, and recursion formulas are developed for computing the functions. By means of a table of these functions for successive primes, it is possible to deplete by a sieve process analogous to the "sieve of Eratosthenes," any series of consecutive integers, so that any integer remaining, if substituted for s will make $6^\kappa \cdot s + 1$ (or $6^\kappa \cdot s - 1$) prime. The paper will be offered to the *Annals of Mathematics.*

7. In the congruence $x^n + y^n + z^n \equiv 0 \pmod{p}$, considered by Professor Dickson, n and p are assumed to be odd prime numbers. As there are obviously solutions prime to p when $p - 1$ is not divisible by n and when $p - 1$ is a multiple of $3n$, it is assumed that $p = mn + 1$, where m is not divisible by 3. In the second supplement to his Théorie des Nombres, Paris, 1825, Legendre has treated the cases $m = 2, 4, 8, 10, 14, 16$, each by a separate discussion; but overlooked the exceptional character of $n = 3$ when $m = 10$ or 14. Professor Dickson develops a method which applies, without a separation of cases, to the values of m considered by Legendre, and obtains the following result. The congruence has no set of solutions prime to p when n and $p = mn + 1$ are odd primes and $m = 2, 4, 8, 10, 14, 16, 20, 22, 26, 28, 32, 40, 56, 64$, apart from the following exceptions: $m = 10, 14, 20$, $n = 3$; $m = 22$, $n = 3, 31$; $m = 26$, $n = 3, 5$; $m = 28$, $n = 7$; $m = 32$, $n = 3$; $m = 40$,

$n=7$; $m=56$, $n=5$, 11, 17, 113, 227; $m=64$, $n=3$, 7, 229, 337, 757. The results are applied to Fermat's last theorem to show that $x^n+y^n+z^n=0$ has no integral solutions, each not divisible by n, for n an odd prime < 1700, the previous limits being 100 (Sophie Germain), 200 (Legendre), 223 (Maillet), 257 (Mirimanoff). The method is due to Sophie Germain and requires the determination of a prime $p=mn+1$ for which the above congruence has no solutions prime to p and such that p is not a divisor of m^m-1. The latter number was completely factored by Legendre for the values m which he treated. Part of this work may be avoided. For example, if m is a multiple of 4, the quadratic reciprocity theorem shows that a prime divisor $mn+1$ (n odd) of m^m-1 must divide $m^{\frac{1}{2}m}-1$. Additional reduction of the exponent may be made when m is a power of 2.

In the second part of the paper, n is given a fixed value and it is proposed to determine the primes $p=mn+1$ for which the congruence has no integral solutions prime to p. A general method of attack is developed and the analysis completed for the cases $n=3$, 5, 7, the number of values of p being 2, 4, 4 respectively.

The first part has been offered for publication in the *Messenger of Mathematics* and the second part in *Crelle's Journal.*

8. In Professor Westlund's note it is proved that the equation $x^n+y^n=nz^n$ is not solvable in rational integers if n is any odd prime and z is not divisible by n.

9. The orbits of particles moving under central forces varying as the first power and as the inverse second, third, and fifth powers have been discussed in detail by many writers, and other special powers also have received attention. The paper of Mr. Hodge and Mr. Moulton considers certain general characteristics of orbits for forces varying as any integral power of the distance, by discussing a transformation of the energy integral expressed in polar coordinates. Without knowing the equations of the orbits, the existence of apses is shown, the aphelian and perihelian distances are found, and certain other properties are deduced.

It is found that for attractive forces the law of the inverse second power gives an intermediate case between two general classes of orbits; the one where the orbits pass through either the center of force or infinity; the other where the orbits have finite perihelion and aphelion distances greater than zero. The paper will be published in the *American Mathematical Monthly.*

10. The totality of the invariant operators of any group G constitutes a characteristic subgroup K known as the central, or the cogredient subgroup of G. When G is abelian, it coincides with K, and when G contains only one invariant operator, K is identity. In these extreme cases the concept of a central does not simplify the considerations with respect to G, but in all other cases it is useful. The quotient group of G with respect to K is the group of cogredient isomorphisms of G and is represented by I_1. The main object of Professor Miller's paper is to determine some of the properties of G from known properties of I_1. The following theorems are proved: If I_1 involves an invariant operator of order n, G contains an invariant commutator of order n and it also has a cyclic quotient group of this order. Moreover, the order of G is divisible by n^3, and I_1 involves two independent operators of order n. The number of Sylow subgroups of any order in G is exactly the same as the number of the Sylow subgroups of the corresponding order in I_1. When all the Sylow subgroups of I_1 are cyclic, every abelian subgroup of I_1 corresponds to an abelian subgroup of G and vice versa. If the order of such an I_1 is $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$, where p_1, $p_2, \cdots, p_m$ are distinct primes in descending order of magnitude, and if I_1 contains a cyclic subgroup of order $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_\gamma^{\alpha_\gamma}$, then G contains a characteristic cyclic subgroup of the same order. The paper will be offered for publication in the *Transactions.*

11. For any surface S referred to lines of curvature the equations of Gauss and Codazzi may be written in the form

$$\frac{\partial M}{\partial v} - \frac{\partial N}{\partial u} = PQ, \quad \frac{\partial P}{\partial v} + MQ = 0, \quad \frac{\partial Q}{\partial u} - NP = 0,$$

where the expression for the corresponding linear element on the gaussian sphere is

$$\overline{d\sigma}^2 = P^2du^2 + Q^2dv^2.$$

Any other equation involving one or more of the functions, taken with the three equations above, determines a certain system of lines on the sphere. We may say therefore that this new equation is characteristic of these lines, or rather of the system of surfaces which have them as the spherical representation of their lines of curvature.

From this standpoint Dr. Young has considered the systems of surfaces defined by the following equations:

$$(1)\ M=0,\quad (2)\ M+N=0,\quad (3)\ \frac{\partial M}{\partial u}+\frac{\partial N}{\partial v}=0,$$

$$(4)\ \frac{\partial^2 \log M}{\partial u\,\partial v}+MN=0,\quad (5)\ \frac{\partial^2 \log MK}{\partial u\,\partial v}-K=0,$$

where

$$K=-\frac{\partial^2 \log M}{\partial u\,\partial v}-MN,\ \text{etc.}$$

12. In this paper Dr. Lunn shows how von Seidel's paraxial equations for a centered optical instrument may be interpreted as defining a two-parameter transformation which generates a three-parameter group of Lie's first type. It is shown that all invariants of this group are functions of the one which occurs in a known formula due to Lagrange and Helmholtz.

13. In his second paper Dr. Lunn points out that the rectilinear oscillation of a particle is determined as simple harmonic by the condition that the dissipation of energy under a resistance proportional to the velocity shall be a minimum when the period and mean square displacement are assigned. The conjugate points bounding the interval of true minimum are determined.

14. Starting with Maxwell's definitions of volume and surface density of charge, and of density of energy, in terms of electric displacement and strength of field, Dr. Lunn gives in this paper a deduction of the conditions of electrostatic equilibrium without assuming the existence of the potential function, which on the contrary appears as a lagrangian indeterminate multiplier.

15. Mr. Schweitzer's note refers to a remark of Enriques in the Encyclopädie der Mathematischen Wissenschaften, volume III, 1, page 34. For the definition of euclidean geometry on the basis of the descriptive-projective geometry, we distinguish, in the well-known manner, between the proper and improper points of the space and agree that all improper points shall lie on the unique improper plane. Every elliptic polar system Σ_0 in this improper plane gives rise to a euclidean geometry $G(\Sigma_0)$.

Thus by means of the class of elliptic polar systems $\{\Sigma_0\}$ in the improper plane is defined a unique class of euclidean geometries $\{G(\Sigma_0)\}$. The members of this latter class are, however, given by the geometry function $G(\Sigma)$ of a variable elliptic polar system Σ in the improper plane. This geometry function is then uniquely defined and consists of a complete set of euclidean propositions in which the congruence propositions appear as implications, having as terms propositional functions of the variable Σ. (Compare Russell, The Principles of Mathematics, pages 13, 82.)

16. As formal types of relational, class, and operational propositions Mr. Schweitzer takes, aRb, $a\epsilon C(b)$, $O_b(a)$. These propositions may be considered as expressing, respectively, "a possesses the relation R with reference to b," "a is in the class C with reference to b," "the operation O with reference to b affects a." It is assumed that aRb implies $a\epsilon C(b)$ implies $O_b(a)$. The following analysis is studied:

$$aRb = aR + b = a + Rb,$$

$$a\epsilon C(b) = aC + b = a + Cb,$$

$$O_b(a) = bO + a = b + Oa,$$

where $aR \neq Ra$, etc., and the symbol of composition "$+$" is an indefinable which implies no ordering of the terms.

17. In his previous papers on the general projective theory of surfaces, Professor Wilczynski assumed that the surface was referred to its asymptotic curves. The object of the present memoir is to find the general expressions for the invariants and covariants if the parameters to which the surface is referred are arbitrary. In this way it becomes possible to apply the results of the author, as developed in his earlier papers, to a given surface without first determining its asymptotic curves. The paper will be offered for publication in the *Transactions*.

18. Dr. Birkhoff effects a classification of irregular integrals by proving that all such integrals are linearly expressible in terms of certain canonical irregular integrals; the coefficients in the linear form are analytic at the singular point $x = a$ in question. These canonical irregular integrals are of the form $x^\kappa\phi(x)$, where $\phi(x)$ is analytic except at $x = a$, and are fully

characterized by the specification of a certain finite set of constants. By means of a method wherein certain integral equations are used, it is found possible to make a direct study of the general irregular integrals whose characteristic properties depend on the aforesaid constants. With the aid of these results it is finally proved that with every irregular integral are associated certain functional equations. Such a functional equation for e^x is $e^{x+c} = e^c \cdot e^x$.

19. The differential calculus furnishes an easy method of finding the general tangent to any curve whose equation is in cartesian coordinates. In this paper Professor Carmichael obtains the same formula for algebraic curves without the aid of the calculus, and extends this result so as to apply to a large number of cases in which the curve is transcendental. The reasoning is of such elementary nature as to be readily understood by a student in his first course. And it is believed that this general formula may be introduced to the pupil's notice with profit. The paper will be offered for publication in the *American Mathematical Monthly.*

20. In his second paper Professor Carmichael points out the form of the cartesian equation of plane algebraic curves symmetrical with respect to each of two rectangular axes, and classifies such curves of the fourth degree, the grouping in classes being determined by certain geometric properties common to those of each class. The paper will be offered for publication in the *American Mathematical Monthly.*

21. In a recent article in the *Transactions of the American Mathematical Society,* Professor Kellogg found it necessary to borrow two facts from the geometry of continuously turning curves; his note presents the proofs of these facts. It will be published in the *American Mathematical Monthly.*

22. In his paper in the *Transactions,* volume 6 (1905), page 504, Professor Loewy transforms any linear group G into a group the matrix of whose general substitution is a compound matrix with square matrices A_{11}, A_{22}, $\cdots$, $A_{\mu\mu}$ in the main diagonal, and with zero matrices to the right of the diagonal; further, the totality of matrices A_{11} form a (maximal) completely reducible group, the A_{22} a completely reducible group, etc. This normalization may be accomplished by a transfor-

mation within a given domain Ω containing the coefficients of the substitutions of G. The groups A_{11}, A_{22}, $\cdots$ completely reducible in Ω, are uniquely determined by G and Ω, when equivalent (conjugate) groups are considered to be identical. The paper by Dr. Schur shows that, when equivalent groups are considered to be not distinct, the groups A_{11}, A_{22}, $\cdots$ are entirely independent of the choice of the domain Ω, so that, if A'_{11}, A'_{22}, $\cdots$, $A'_{\mu'\mu'}$ are the analogous groups for a new domain Ω' likewise containing the coefficients of the substitutions of G, then $\mu' = \mu$ and A'_{ii} is equivalent to A_{ii}. Of the various additional theorems established by Dr. Schur, mention may be made of the following: Every linear group G irreducible in a domain Ω is completely reducible in the domain Z of all numbers; if the most general matrix commutative with G has exactly r distinct characteristic roots, G decomposes in Z into r irreducible groups F, F_1, $\cdots$ of equal degree; these r groups may be represented in r conjugate algebraic domains $\Omega(\rho_i)$. If exactly l of these r groups are non-equivalent, then r/l is an integer which divides the degree of the group F. The paper will be offered for publication in the *Transactions*.

23. The character of the solutions of homogeneous linear differential equations with periodic coefficients (period 2π) is well known from the writings of Floquet, Callandreau, and others. The purpose of Mr. MacMillan's paper is to exhibit similarly the character of the solutions when the equations are linear with periodic coefficients but non-homogeneous. It is found that, when the characteristic exponents are all distinct and none of them congruent to zero (mod $\sqrt{-1}$) and the non-homogeneous terms have the period 2π, the particular solutions are periodic with the period 2π. If one of the characteristic exponents is congruent to zero (mod $\sqrt{-1}$), then in general the particular solution will contain terms involving the independent variable as a linear factor, that is to say the solution is not periodic.

If the non-homogeneous terms of the differential equations contain terms of the form $\sin (i + \lambda)t$ or $\cos (i + \lambda)t$ (where i is an integer, and λ is any real number) then the particular solution depending upon these terms is trigonometric, provided the characteristic exponents are all distinct and not congruent to zero nor equal to $\lambda\sqrt{-1}$. When these conditions are not fulfilled the solution in general is not periodic. The appropriate discussion for these cases is given.

24. In the *Mathematische Annalen* volume 12, page 376, Grassmann has defined the middle product of two vectors as a "formal" sum of their outer and inner products, namely, $AB = [AB] + \lambda[A/B]$, the product being associative if $\lambda = -1$ and non-associative if $\lambda = +1$. Mr. Schweitzer observes that the quaternion and its extension occur as operators in the grassmannian vector space of 2^n dimensions ($n = 1, 2, 3, \cdots$). For example, for $n = 2$, if

$$X = x_1E_1 + x_2E_2 + x_3E_3 + x_4E_4,$$

the vectors $i'x$, $j'x$, $k'x$ are suitably defined and the vectors $i'/\lambda' \cdot X$, $j'/\lambda'' \cdot X$, $k'/\lambda''' \cdot X$, where λ', λ'', λ''' are constants different from zero, are determined by the specification that the outer product $X \cdot i'X \cdot j'X \cdot k'X$ shall be identically equal to $\rho(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2E_1E_2E_3E_4$, where ρ is a constant. These vectors may also be determined by a simple tactical construction of a square matrix of 2^n elements. It is found then that as operators on the vector X,

$$\left(\frac{i'}{\lambda'}\right)^2 = \left(\frac{j'}{\lambda''}\right)^2 = \left(\frac{k'}{\lambda'''}\right)^2 = -1, \quad \frac{i'}{\lambda'} \cdot \frac{j'}{\lambda''} = -\epsilon\frac{k'}{\lambda'''},$$

$$\frac{i'}{\lambda'} \cdot \frac{j'}{\lambda''} \cdot \frac{k'}{\lambda'''} = \epsilon,$$

where $\epsilon^2 = 1$. The quaternion Q is defined as an operator $Q(X', X)$ which transforms a grassmannian vector X into the vector X'. For the fundamental unit vectors E_1, E_2, E_3, E_4 the following relations are valid: (1) If $E_{r_1}|E_{r_2}$ is the grassmannian complement of $E_{r_3}E_{r_4}$, then $Q(E_{r_1}E_{r_2}) = \epsilon Q(E_{r_3}E_{r_4})$. (2) If E_{r_1}, E_{r_2}, E_{r_3} are any three unit vectors, then

$$Q(E_{r_1}E_{r_2}) \cdot Q(E_{r_2}E_{r_3}) = Q(E_{r_1}E_{r_3}).$$

(3) If $E_{r_1} \neq E_{r_2}$, then $Q(E_{r_1}E_{r_2}) = -Q(E_{r_2}E_{r_1})$. The generalization of these relations is discussed.

Thus, from the operative viewpoint, the quaternion appears in a real and homogeneous manner; also what may be called the operative sum and product of two grassmannian vectors in 2^n space is readily defined.

H. E. SLAUGHT,
Secretary of the Section.

HEINRICH MASCHKE.

Resolutions on the Death of Professor Heinrich Maschke.

Whereas, in the death of Professor Heinrich Maschke the Chicago Section of the American Mathematical Society suffers the loss of one of its most honored, influential, and beloved members, the undersigned committee appointed by the Section hereby expresses its deep appreciation of the services and character of Professor Maschke.

From the first organization of the Chicago Section until the present session Professor Maschke has been one of its most active and inspiring members. By his genial qualities, his unusual sympathy as a teacher, his integrity and intellectual honesty, he has won and held the affection of those who have known him. By his ability as an investigator he has contributed greatly to the development of productive mathematical scholarship in the formative period of the Society, and in his own person he has exemplified the influence of German scholarship which has contributed so potently to this development. In the death of Professor Maschke the Section for the first time feels the loss of one of its leaders.

The committee recommends that this resolution be incorporated in the records of the Section and that a copy be transmitted to Mrs. Maschke as an expression of our sympathy in her deep loss.

E. B. Van Vleck,
Alexander Ziwet,
H. E. Slaught.

CRITERIA FOR THE IRREDUCIBILITY OF A RECIPROCAL EQUATION.

BY PROFESSOR L. E. DICKSON.

(Read before the Chicago Section of the American Mathematical Society, April 17, 1908.)

1. A reciprocal equation $f(x) = x^m + \cdots = 0$ is one for which

$$x^m f(1/x) \equiv cf(x).$$

Replacing x by $1/x$, we see that $f \equiv c^2 f$, $c = \pm 1$. Now $f(x)$ has the factor $x \pm 1$ and hence is reducible, unless m is even and $c = +1$. Further discussion may therefore be limited to equations

$$(1)\quad F(x) \equiv x^{2n} + c_1 x^{2n-1} + c_2 x^{2n-2} + \cdots + c_2 x^2 + c_1 x + 1 = 0$$

of even degree and having

$$(2)\qquad x^{2n} F(1/x) \equiv F(x).$$

Let R be a domain of rationality containing the c's.

Under the substitution

$$(3)\qquad x + 1/x = y,$$

$x^{-n}F(x)$ becomes a polynomial in y,

$$(4)\qquad \phi(y) = y^n + k_1 y^{n-1} + \cdots + k_n,$$

with coefficients in R. By a suitable choice of the c's, the k's may be made equal to any assigned values.

We shall establish in §§ 2–7 the following:

THEOREM. *Necessary and sufficient conditions for the irreducibility of $F(x)$ in the domain R are*

(I) *$\phi(y)$ must be irreducible in R.*

(II) *$F(x)$ must not equal a product of two distinct irreducible functions of degree n.*

The second condition is discussed in §§ 8–10.

2. The irreducibility of $F(x)$ in R implies that of $\phi(y)$. For, if

$$\phi(y) = (y^l + \cdots)(y^m + \cdots), \quad (l + m = n, \quad l > 0, \quad m > 0),$$

then would

$$F(x) = x^l\{(x + 1/x)^l + \cdots\} \cdot x^m\{(x + 1/x)^m + \cdots\}.$$

3. If $F(x)$ has in R an irreducible factor

$$A(x) = x^{2r+1} + a_1x^{2r} + \cdots + a_{2r}x + a_{2r+1}$$

of odd degree, then $F(x)$ has the irreducible factor.

$$B(x) \equiv \frac{x^{2r+1}}{a_{2r+1}} A\left(\frac{1}{x}\right) = x^{2r+1} + \frac{a_{2r}}{a_{2r+1}} x^{2r} + \cdots + \frac{a_1}{a_{2r+1}} x + \frac{1}{a_{2r+1}},$$

not identical with $A(x)$. For, from $F \equiv AQ$ and (2) follows

$$F(x) \equiv B(x)Q', \quad Q' \equiv a_{2r+1}x^{2n-2r-1}Q(1/x).$$

Next, if $B \equiv A$, then

$$\begin{aligned} A &= x^{2r+1} \pm 1 + a_1x(x^{2r-1} \pm 1) \\ &\qquad + a_2x^2(x^{2r-3} \pm 1) + \cdots + a_rx^r(x \pm 1), \end{aligned}$$

so that A would have the factor $x \pm 1$ and be reducible.

4. If $\phi(y)$ is irreducible in R, $F(x)$ has in R no irreducible factor A of odd degree $< n$. For, if so, $P \equiv AB$, where B is given in § 3, would be a self-reciprocal factor of $F(x)$. In fact,

$$P(1/x) = A(1/x)A(x)/x^{2r+1}a_{2r+1}, \quad x^{2(2r+1)}P(1/x) = AB = P(x).$$

Hence, in view of (3), $x^{-(2r+1)}P(x)$ would equal a factor of degree $2r + 1$ of $\phi(y)$.

5. If $F(x)$ has in R an irreducible factor

$$A(x) = x^{2r} + a_1x^{2r-1} + \cdots + a_{2r-1}x + a_{2r}$$

of even degree, then $F(x)$ has the irreducible factor

$$B(x) = \frac{1}{a_{2r}} x^{2r} A\left(\frac{1}{x}\right).$$

If $B(x) \equiv A(x)$, A is self-reciprocal, viz.,

$$A(x) = x^{2r} + 1 + a_1(x^{2r-1} + x) + \cdots + a_{r-1}(x^{r+1} + x^{r-1}) + a_rx^r.$$

In fact, the conditions for $B \equiv A$ are

$$a_{2r} = \pm 1, \quad a_{2r-1} = \pm a_1, \quad a_{2r-2} = \pm a_2, \cdots.$$

For the lower signs, A has the factor $x^2 - 1$, contrary to hypothesis.

6. If $\phi(y)$ is irreducible in R, $F(x)$ has in R no irreducible factor A of even degree $< n$. For, by § 5, either B is distinct from A so that AB is a self-reciprocal factor of $F(x)$, or else A itself is a self-reciprocal factor. In either case $\phi(y)$ would have in R a factor of degree $< n$.

7. It follows from §§ 4, 6 that, when $\phi(y)$ is irreducible in R, $F(x)$ has no irreducible factor of degree $< n$. Further, by §§ 3, 5, an irreducible factor $A(x)$ of degree n implies a second irreducible factor $x^nA(1/x)$, algebraically distinct from $A(x)$. The theorem of § 1 is therefore proved.

8. It remains to consider the case $F = AB$,

$$A = x^n + a_1x^{n-1} + \cdots + a_n,$$

$$B = x^n + \frac{a_{n-1}}{a_n}x^{n-1} + \cdots + \frac{a_1}{a_n}x + \frac{1}{a_n},$$

where A and B are distinct irreducible functions in R. To determine the a_i we have n distinct relations

$$(5) \quad a_1 + a_{n-1}/a_n = c_1, \quad a_2 + (a_1a_{n-1} + a_{n-2})/a_n = c_2, \cdots$$

We may eliminate $a_1, \cdots, a_{n-1}$ and obtain an equation for a_n. As shown in § 9, this equation is of degree 2^n. Except for certain sets of values of the c_i, we may express $a_1, \cdots, a_{n-1}$ rationally in terms of a_n; the problem is then reduced to the consideration of the rationality of a root of the equation of degree 2^n. This equation for a_n is a reciprocal equation. In fact, if we set

$$A_1 = a_{n-1}/a_n, \quad \cdots, \quad A_{n-1} = a_1/a_n, \quad A_n = 1/a_n,$$

equations (5) become

$$(5') \; A_1 + A_{n-1}/A_n = c_1, \; A_2 + (A_1A_{n-1} + A_{n-2})/A_n = c_2, \cdots.$$

That the equations (5′) are throughout of the same form as equations (5) is evident from the fact that we have merely interchanged the rôles of the factors A and B of F. Hence the equation in a_n, obtained by eliminating $a_1, \cdots, a_{n-1}$ from

(5), is identical with the equation in $A_n = 1/a_n$, obtained from (5′).

9. Denote the roots of $F = 0$ by

(6) $$\alpha_1, \quad \alpha_1^{-1}, \quad \alpha_2, \quad \alpha_2^{-1}, \quad \cdots, \quad \alpha_n, \quad \alpha_n^{-1}.$$

A factorization $F = AB$, of the kind considered in § 8, corresponds uniquely to a separation of the roots (6) into two sets each of n roots, such that reciprocal roots belong to different sets. Hence the roots of the first set may be selected in

$$\frac{2n(2n-2)(2n-4)\cdots 2}{n!} = 2^n$$

ways. The number of factors A is thus 2^n.

10. For $n = 2$, we set $\alpha_1 = \alpha$, $\alpha_2 = \beta$, and have

(7) $$\alpha+\beta+\alpha^{-1}+\beta^{-1}=c_1, \quad 2+\alpha\beta^{-1}+\beta\alpha^{-1}+\alpha\beta+\alpha^{-1}\beta^{-1}= c_2.$$

From these we derive

$$\alpha^2 + \beta^2 + \alpha^{-2} + \beta^{-2} = c_1^2 - 2c_2.$$

Hence $\alpha\beta + \alpha^{-1}\beta^{-1}$ and $\alpha\beta^{-1} + \alpha^{-1}\beta$ are the roots of

(8) $$z^2 - (c_2 - 2)z + c_1^2 - 2c_2 = 0.$$

The quartic for a_2(§ 8) is obtained by setting

(9) $$z = a_2 + a_2^{-1}.$$

By (7_1), $\alpha + \beta$ is a rational function of $\alpha\beta$ and c_1 when $c_1 \neq 0$. Hence, for $c_1 \neq 0$, the necessary and sufficient conditions for the factorization $F = AB$ in R are that the roots

(10) $$z_\pm = \tfrac{1}{2}(c_2 - 2) \pm [(1 + \tfrac{1}{2}c_2)^2 - c_1^2]^{1/2}$$

of (8) be rational and that one of the values $(z_\pm^2 - 4)^{1/2}$ be rational, so that (9) shall lead to a rational value of a_2. Incorporating the condition that (4) shall be irreducible in R, we obtain the

THEOREM.* *The necessary and sufficient conditions that*

$$x^4 + c_1x^3 + c_2x^2 + c_1x + 1 \quad (c_1 \neq 0)$$

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* For other proofs by the writer, see *Amer. Math. Monthly*, vol. 10 (1903), p. 221; vol. 15 (1908), p. 75. The first paper cited also treats reciprocal sextic equations.

shall be irreducible in a domain R are that $(c_1^2 - 4c_2 + 8)^{1/2}$ *be irrational, and that either* $l = [(1 + \frac{1}{2}c_2)^2 - c_1^2]^{1/2}$ *be irrational or else l rational and* $[\frac{1}{2}c_2^2 - c_1^2 - 2 \pm (c_2 - 2)l]^{1/2}$ *both irrational.*

11. The only linear fractional transformations which replace a reciprocal equation by a reciprocal equation are

$$(11) \qquad x' = \pm \frac{\alpha x + \beta}{\beta x + \alpha} \qquad (\alpha^2 \neq \beta^2).$$

Then y, given by (3), undergoes the transformation

$$(12) \qquad y' = \pm \frac{(\alpha^2 + \beta^2)y + 4\alpha\beta}{\alpha\beta y + \alpha^2 + \beta^2}.$$

The transformation on $\frac{1}{2}y$ is the square of (11).

UNIVERSITY OF CHICAGO,
March, 1908.

A NEW GRAPHICAL METHOD FOR QUATERNIONS.

BY PROFESSOR JAMES BYRNIE SHAW.

(Read before the Southwestern Section of the American Mathematical Society, November 30, 1907.)

1. ANY quaternion q may be written in the form $q = (w + xi) + (y + zi)j$. For convenience let us represent numbers of the form $w + xi$ (practically equivalent to ordinary complex numbers save in their products by j) by Greek characters, so that q may be written

$$q = \alpha + \beta j,$$

where for any number β we have $\beta j = j\bar{\beta}$, $\bar{\beta}$ being the conjugate of β.

The tensor of q is then the square root of the sum of the squares of the moduli of α, β. Also the scalar of q is $\frac{1}{2}(\alpha + \bar{\alpha})$, that is, the real part of α.

2. The product of $q = \alpha + \beta j$ and $r = \gamma + \delta j$ is

$$qr = (\alpha\gamma - \beta\bar{\delta}) + (\alpha\delta + \beta\bar{\gamma})j,$$

and also we have

$$rq = (\alpha\gamma - \bar{\beta}\delta) + (\bar{\alpha}\delta + \beta\gamma)j.$$

The sum of q and r is $q + r = \alpha + \gamma + (\beta + \delta)j$.

3. The graphical representation is as follows: Assume two independent planes, the A-plane and the B-plane, on which we represent α and β like ordinary complex numbers. These representations are the A and B projections of q, respectively. The sum of q and r has for its projections the sums of the respective projections of q and r. To represent the projections of the product qr, we find by the usual methods the products $\alpha\gamma$, $\beta\bar{\delta}$, $\alpha\delta$, $\beta\bar{\gamma}$. To find $\alpha\gamma$, γ is turned through the angle of α, then extended in the ratio $T\alpha:1$. So for the others. We then add vectorially $\alpha\gamma$ and $-\beta\bar{\delta}$; $\alpha\delta$ and $\beta\bar{\gamma}$.

4. If we consider r as fixed, and let the multiplier q be such that the extremities of its projections describe straight lines, then the extremities of the projections of qr will describe straight lines. For if

$$\alpha = \lambda_1 + t(\lambda_2 - \lambda_1), \quad \beta = \mu_1 + t(\mu_2 - \mu_1),$$

then

$$qr = [\lambda_1\gamma + t(\lambda_2 - \lambda_1)\gamma - \mu_1\bar{\delta} - t(\mu_2 - \mu_1)\bar{\delta}] + [\lambda_1\delta + t(\lambda_2 - \lambda_1)\delta + \mu_1\bar{\gamma} + t(\mu_2 - \mu_1)\bar{\gamma}]j.$$

The extremity of the A projection of qr thus describes the line

$$\rho = \lambda_1\gamma - \mu_1\bar{\delta} + t[(\lambda_2 - \lambda_1)\gamma - (\mu_2 - \mu_1)\bar{\delta}],$$

which passes through the points $\lambda_1\gamma - \mu_1\bar{\delta}$ and $\lambda_2\gamma - \mu_2\bar{\delta}$. The extremity of the B projection of qr describes the line

$$\rho = \lambda_1\delta + \mu_1\bar{\gamma} + t[(\lambda_2 - \lambda_1)\delta + (\mu_2 - \mu_1)\bar{\gamma}],$$

which passes through the points $\lambda_1\delta + \mu_1\bar{\gamma}$ and $\lambda_2\delta + \mu_2\gamma$. Further if $\alpha = e^{i\omega t}\alpha_1$, $\beta = e^{i\omega t}\beta_1$, then

$$qr = e^{i\omega t}(\alpha_1\gamma - \beta_1\bar{\delta}) + e^{i\omega t}(\alpha_1\delta + \beta_1\gamma)j.$$

Thus the projections of the product will describe circles with the same period as those of q.

5. We have $S.q\bar{r} = 0$ when $S(\alpha\bar{\gamma} + \beta\delta) = 0$, that is

$$\alpha\bar{\gamma} + \beta\delta = ti.$$

Solving for β we have for any value of α,

$$\beta = -\frac{\alpha\bar{\gamma}}{\delta} + t\frac{i}{\delta}.$$

That is, for any value of α the values of β terminate on a straight line perpendicular to $\bar{\delta}$. Likewise for any value of β the values of α terminate on the line

$$\alpha = -\frac{\beta\delta}{\bar{\gamma}} + t\frac{i}{\bar{\gamma}},$$

which is perpendicular to γ. This is equivalent to saying that if β terminates on the line through ξ at right angles to $\bar{\delta}$; α will terminate on the line through $-\xi\delta/\bar{\gamma}$ perpendicular to γ. Or in brief, to the line through θ/δ perpendicular to $\bar{\delta}$ corresponds the line through $-\theta/\bar{\gamma}$ perpendicular to γ, in the sense that any quaternion q whose projections terminate on these two lines in the A and B planes respectively, is *perpendicular* to the quaternion $r = \gamma + \delta j$.

6. The applications of this method to four-dimensional space are obvious, the A and B planes being two planes having only one point in common, the origin. The representation is in fact an adaptation of the descriptive geometry of four-dimensional space to the representation of quaternions as four-dimensional vectors, although this interpretation is not essential to the representation.

LOGIC AND THE CONTINUUM.

BY PROFESSOR EDWIN BIDWELL WILSON.

THE problem whether every set and in particular the continuum can be well ordered has attracted considerable attention since the days when G. Cantor first stated it in 1883.* In 1904 Zermelo offered an affirmative solution of the problem,† but his solution has not been generally regarded with favor and the discussion of the whole problem has now turned largely to a discussion of his solution. In a recent article he has summarized and discussed this discussion so fully that no repetition is called for at this time.‡ In entering so vast an arena of conflict, I would make no pretense of settling the difficulties or of solving the problem; I merely desire to call attention to some general matters of logic which have an application to this problem and to others, and which appear to me to have been somewhat neglected. The introduction of a different point of view may be of service. My comments will at first be directed upon the idea of the categoricity or sufficiency of a set of postulates.

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* *Mathematische Annalen*, vol. 21, p. 550.

† Zermelo, "Beweis, dass jede Menge wohlgeordnet werden kann," *Mathematische Annalen*, vol. 59, pp. 514–516.

‡ Zermelo, "Neuer Beweis für die Möglichkeit einer Wohlordnung," *ibid.*, vol. 65 (1907), pp. 107–128.

Although the method of nominal definition from logical elements is regarded in many quarters as preferable to the determination by means of postulates, when a specified field of mathematics is to be investigated, there still remains so much difficulty in the way of the purely nominal definition of mathematics that the determination by means of postulates cannot as yet be considered as obsolete, perhaps not even as obsolescent; and this method apparently must obtain in laying the foundations of logic itself, whether or not the nominal definition is finally adopted for the subsidiary sciences. The ideas that a set of postulates should not be redundant and must not be contradictory have been widely exploited. There is, however, another idea which so far as I am aware was first clearly stated and emphasized by Huntington and which has not yet received a great measure of recognition. This idea is that a set of postulates may be sufficient or categorical.*

A set of postulates $[P] = P_1, P_2, \cdots, P_n$ connecting a set of undefined symbols $[S] = S_1, S_2, \cdots, S_m$ is said to be categorical if between the elements of any two assemblages, each of which independently contains the undefined symbols $[S]$ and satisfies the postulates $[P]$, it is possible to set up a one to one correspondence which preserves the significance of the undefined symbols. This last restriction is vital — it ensures a sort of logical conformality in the correspondence. For it is ob-

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*Huntington, "A complete set of postulates for the theory of absolute continuous magnitude, *Transactions of the American Mathematical Society*, vol. 3 (1902), pp. 264–279. "Complete sets of postulates for the theories of positive integral and positive rational numbers," *ibid.*, pp. 280–284. "Complete sets of postulates for the theory of real quantity," *ibid.*, vol. 4 (1903), pp. 358–370. In these earlier articles Huntington uses the term sufficient; in later communications, "A set of postulates for real algebra, comprising postulates for a one-dimensional continuum and for the theory of groups," *ibid.*, vol. 6 (1905), pp. 17–41, and "The fundamental laws of addition and multiplication in elementary algebra," *Annals of Mathematics*, ser. 2, vol. 8 (1906), pp. 1–44, he adopts somewhat hesitatingly the term categorical from a paper of Veblen's and points out, *Annals*, p. 26, that Peano had and apparently realized that he had a categorical set of postulates for positive integers as early as 1891.

vious that if any set of elements $[K]$ which satisfies the postulates $[P]$ on the symbols $[S]$ may be put into a one to one relation with a set of elements $[K']$, it must be possible to interpret $[P]$ and $[S]$ in the set $[K']$ by a mere transference from the original set $[K]$. This procedure is of frequent occurrence in mathematics,* but it has little to do with categoricity for the reason that $[K]$ and $[K']$ are not determined independently.

It is not always desirable and indeed not always possible to obtain a set of postulates which shall be categorical: for it may well happen that the systems to be determined are such that not even a one to one correspondence between their elements is available, to say nothing of the preservation of the interpretation of the symbols $[S]$. For instance in defining a group it would clearly be unwise so to restrict the definition that all groups were of equal multitude. And in defining a geometry in a space of constant curvature, there is little to be gained by a restriction which specifies the magnitude of the curvature unless it be zero. It is, however, advisable to have a categorical determination of euclidean geometry and of the continuum and of real algebra. This has been accomplished by Huntington and Veblen.† In case these domains of mathematics are defined nominally it is equally important that the definition shall be sufficiently detailed to ensure an equivalent precision in the system defined.

Huntington gives a subsidiary definition or explanation of the idea of categoricity wherein he asserts that if a set $[P]$ of postulates on the undefined symbols $[S]$ is categorical, then every proposition concerning $[S]$ must be deducible from the postulates $[P]$ or be in contradiction with them.‡ In interpreting this statement it should be noted that not every conceivable property of the defined set $[K]$ is asserted as either deducible from $[P]$ or in contradiction with $[P]$, but only such properties as are expressible solely in terms of $[S]$. Thus the statement that the rational numbers greater than 0 are red and those less than 0 are blue is neither in contradiction with

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* A little reflection will show that even here the value of the transference of properties from $[K]$ to $[K']$ is almost nil except as the transferred properties may be related to properties which $[K']$ independently possesses.

† References to Huntington have been given. Veblen, "A system of axioms for geometry," *Transactions of the American Mathematical Society*, vol. 5 (1904), pp. 343–384.

‡ See his last paper cited, or Veblen's.

Huntington's set of postulates nor deducible from them, because the statement contains terms (red, blue) which are not defined or nominally definable in the system; moreover, any statement about "alternate rational numbers," where alternate refers to the numbers in their natural order, is neither compatible nor incompatible with the postulates, because that statement is not a proposition at all but merely a meaningless collocation of words.*

I have advisedly used the terms compatible with $[P]$ and incompatible with $[P]$ in the respective places of deducible from $[P]$ and in contradiction with $[P]$. Is this a justifiable change? It is readily seen that any proposition (statement which may be either true or false but not meaningless) phrased in terms of $[S]$ or derived symbols must be either compatible with $[P]$ or incompatible with $[P]$: for, owing to the categoricity of the determination, it is impossible that the proposition should be true of one set $[K]$ and false of another set $[K']$ both of which satisfy the postulates. Does it follow that every such proposition must either be deducible from $[P]$ or in contradiction with $[P]$? This question, this suggestion that compatibility and deducibility may not be the same when applied to categorically determined systems is vital in logic and requires careful discussion.†

In the first place it is evident that if the determination of a system is not categorical, then compatibility and deducibility cannot be entirely equivalent. This follows from the very idea of compatibility, to wit, that any proposition phrased in terms of the undefined symbols shall either be true for all the systems determined by $[P]$ or be false for them all. Any individual postulate of a set $[P]$ which is not redundant is neither compatible nor incompatible (or is both compatible and incompatible, if preferred) with the other postulates of [P].* In the second place consider a system determined categorically. Here every proposition phrased in terms of [S] is either compatible or incompatible with [P]. What, however, does the word deducible mean? The meaning is entirely relative to the system of logic which is available for drawing conclusions from the set of primitive propositions [P]. Some may consider that the human mind has instinctively at its disposal all valid methods of deduction. This is a tremendous postulate, and one entirely devoid of other than sentimental value. In fact, if it leads to the abandoning of the research for valid methods of deduction, it is dangerous and worse than useless. It is an essential of the modern attitude in logic that the deducer should state distinctly his form of inference. Hence deducible cannot be regarded as equivalent to compatible.

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* The difficulties that are involved in meaningless statements have been recognized since very early times; for instance, the case of the liar who says he is a liar. Such is also in all probability the difficulty with the class of all classes; see Zermelo, "Untersuchungen über die Grundlagen der Mengenlehre," *Mathematische Annalen*, vol. 65 (1908), pp 261–281.

† Huntington quotes a remark of Davis on this subject in a later paper "A set of postulates for ordinary complex algebra," *Transactions*, vol. 6 (1905), p. 210, footnote †; Veblen dismisses it with the statement that any additional postulate would have to be considered redundant even were it not deducible from the categorical set by a finite number of syllogisms, *loc. cit.*, p. 346. As I do not know what he means by this, I am unable to say whether or not his position is in practical accord with that which I shall set forth in the following paragraphs.

It is clear that in an ideal perfection of logic compatibility and deducibility would be equivalent for categorically defined systems. That state of perfection appears at present to be very remote. The constant attempt to bring compatibility and deducibility into coincidence will undoubtedly do much to advance the condition of logic, just as in the days before Huntington so clearly stated the idea of categoricity that attempt did much to advance mathematics.† It appears to me, however, that it may be a distinct gain in precision and hence a considerable advantage to admit the following orienting propositions, namely:

So long as there is an unsolved problem of pure mathematics the solution may be lacking

1° because the class of objects to which the problem belongs is not sufficiently determined or

2° because the available logical methods of deduction are insufficient; but in case the class of objects is categorically determined, 2° alone applies.‡

To take some examples. Fermat's problem of the solution of

$$x^n + y^n = z^n, \quad n > 2,$$

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* This statement is capable of a small amount of generalization by including under proper restrictions theorems as well as postulates.

† For instance, the development of noneuclidean geometry could be interpreted as a somewhat unconscious striving in this direction.

‡ There is, of course, the possibility that the lack of solution is due to our failure to perceive the method by which the logical principles already in our possession should be combined to reach the desired result. Logically, however, this is a very different sort of failure from either of those mentioned, and it has been thought better not to refer to it in the text.

in terms of integers cannot fail of solution because of insufficient determination of the class of integers, inasmuch as integral algebra has been established upon a categorical basis. From the nature of this problem it appears likely that the lack of solution is not due to the incompleteness of the methods of deduction, but rather to a failure properly to combine the existing methods. Goldbach's theorem that any even number may be written as the sum of two primes cannot, for a similar reason, fail of proof owing to an insufficiency of determination. Peano states,* if I understand his inflexionless latin, that this theorem fails of a satisfactory demonstration because, like the theorem that any infinite set contains an enumerable subset, it depends on the application of the principle of Zermelo (see below). Thus for Peano and Zermelo deducible has different meanings; and any one is at liberty to side with either party to the dispute.† It may, however, be pointed out that, by some method of combining forms of deduction which both would admit, Goldbach's theorem might perhaps be demonstrated: but it is difficult to see how such a reconciliation could be operated in regard to the theorem on the existence of an enumerable subset in any infinite set.

In discussing the problem of the well-ordering of the continuum from the point of view I have here set forth, it is first necessary to ask if the definition of well-ordering can be phrased in terms of the undefined symbols introduced in the categorical determination of the continuum. On comparing Zermelo's definition of well-ordering and Huntington's set of postulates,‡ it appears that the definition can be so phrased without introducing new symbols other than the general logical constants introduced in framing the postulates. Hence Zermelo has no right to add any new postulate expressing a property of the continuum. As a matter of fact, the postulate that he does add concerns classes in general and the usage he makes of it and his own statements show that what he has done is to demand a new postulate or principle of logic. Thus to prove

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*Peano, "Supra theorema de Cantor-Bernstein," *Revista de Matematica*, vol. 8 (1906), pp. 136–157; see especially p. 146.

† As Zermelo, in common with most other investigators, does not state what his logical postulates are, it is impossible to say just what deducible does mean for him. This difficulty, which is scarcely felt at all in ordinary reasoning, becomes very troublesome when one tries to follow the great number of different writers on transfinites.

‡ Zermelo, *loc. cit.*, *Annalen*, 65, p. 111. Huntington, *loc. cit.*, *Transactions*, 6, p. 31.

that the proposition that the continuum is well ordered is compatible with the postulates concerning the continuum, he enlarges his logic and shows that any set is well ordered.

In view of the facts that it may be doubted whether our logic is yet complete and that Zermelo's postulate is apparently not in contradiction with the other logical postulates, it is difficult to see how any one can deny him the right to proceed as he does. The only question appears to be whether his method is valuable. And certainly, if such an elementary theorem of arithmetic as Goldbach's theorem cannot be proved or disproved without some such addition as Zermelo makes to our logical procedure, it seems as if some such addition were imperative; for however one may hesitate about committing himself to the intricacies of the continuum, he surely would not indefinitely consign Goldbach's theorem to the class of non-deducibles. It therefore is proper to examine the new principle of logic and its consequences.

The first statement of Zermelo's principle is: For an infinite ensemble of sets it is possible to find a correspondence which correlates to each set a single element of that set.* For a finite ensemble of sets the proposition is readily admitted. The same is true of the later formulation of the postulate: A set S which is divided into subsets A, B, C, $\cdots$ each containing at least one element but containing no elements in common contains at least one subset S_1 which has just one element in common with each of the subsets A, B, C, $\cdots$.† To this postulate Peano objects that: one may not apply an infinite number of times an arbitrary law by virtue of which one correlates to a class some member of that class.‡ Here are two postulates by two different authorities; the postulates are contradictory, and each thinker is at liberty to adopt whichever appears to him the more convenient.

Lebesgue has recently pointed out the high degree of unattainability which characterizes the correspondence postulated by Zermelo. § I should like to call attention to another awkwardness of the theory. Consider at first a finite set of elements, say $M(1, 2, 3, 4)$, and follow out the statement of Zermelo's

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* Zermelo, *loc. cit.*, *Annalen*, 59, p. 516.

† Zermelo, *loc. cit.*, *Annalen*, 65, p. 110.

‡ Peano, *loc. cit.*, *Revista*, 8, p. 145. Peano had taken this position in an earlier communication in 1890, *Mathematische Annalen*, vol. 37, p. 210.

§ Lebesgue, "Contribution a l'étude des correspondances de M. Zermelo," *Bulletin de la Société Mathématique de France*, vol. 35, pp. 202–212.

fundamental theorem: Ist durch irgend ein Gesetz jeder nichtverschwindenden Untermenge einer Menge M eines ihrer Elemente als "ausgezeichnetes Element" zugeordnet, so besitzt die Menge $\mathfrak{U}(M)$ aller Untermengen von M eine und nur eine Untermenge $\mathfrak{M}$ von der Beschaffenheit, dass jeder beliebigen Untermenge P von M immer ein und nur ein Element P_0 von $\mathfrak{M}$ entspricht, welches P als Untermenge und ein Element von P als ausgezeichnetes Element enthält. Die Menge M wird durch $\mathfrak{M}$ wohlgeordnet.* To avoid all the difficulties of following the author's deduction of this theorem, let it be assumed that the theorem is properly deduced.

In the first place it is necessary to construct the set $\mathfrak{U}(M)$ of all subsets of M. This is the set

1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 124, 134, 234, 1234.

The number of elements in this set is $4 + 6 + 4 + 1 = 15$. More generally if M had contained n elements, the set $\mathfrak{U}(M)$ would have contained

$$n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots + n + 1 = 2^n - 1$$

elements. Next it is to be supposed that in each of these sets some element has been picked out as ausgezeichnet. This may be accomplished by printing that element of each set in red. The number of ways in which it may be accomplished is 20736. In case the number of elements had been n, the number of ways would have been

$$(a) \qquad 2^{\frac{n(n-1)}{2!}} \cdot 3^{\frac{n(n-1)(n-2)}{3!}} \cdot \cdots \cdot (n-1)^n \cdot n.$$

The next step is the assertion of the existence of the set $\mathfrak{M}$ of the specified properties. This set will depend in large measure upon the particular way in which the ausgezeichnetes element of each set has been chosen. If it were agreed to print the smallest integer in each set in red, the set $\mathfrak{M}$ would be

1234, 234, 34, 4,

and the corresponding way in which M would be well ordered would be 1, 2, 3, 4. However, by the method outlined by Zermelo, loc. cit., page 110, it is clear that with a proper distribution of the red ink it would be possible to obtain a set $\mathbf{M}$ which would give M any one of its possible 4 ! arrangements. In like manner if M contained n elements, a proper selection of the red elements would give rise to an $\mathbf{M}$ which would ensure to M any of its n ! possible orders.

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* Zermelo, *loc. cit.*, *Annalen*, 65, p. 108.

As a matter of fact, it may be seen from Zermelo's reasoning that $\mathbf{M}$ must contain M, and at least one of the subsets of M in which only one element of M is missing, also at least one of the subsets of this subset where just one additional element is missing, and so on. Thus suppose that $\mathbf{M}$ were picked in accordance with this principle as

$$1234,\ 134,\ 13,\ 3.$$

The corresponding ordering of M would be 2, 4, 1, 3, and it would not be hard to assign a red element in each subset of M on this basis. This assignment could probably be made in an enormous variety of ways; for the number of ways in which red elements may be selected is (a), in this case 20736, whereas the possible arrangements for M number only n !, in this case 24.

Thus in the case of the well-ordering of a finite set a very complicated mechanism has been introduced which suffices not only to order the set, but to order it in all possible ways and moreover to accomplish this ordering in an even vastly greater number of ways. One is tempted to say that not only has the author proved his proposition (for finite sets), but so much more than proved it as to render the conception of well-ordering almost worthless. In view of the fact that there are many who do not believe in the transfinite cardinals at all and that those who do believe find it no easy matter to escape error in manipulating these numbers, one must be very cautious in extending to infinite sets a method of reasoning analogous to that here employed with reference to finite sets. Some general considerations may, however, be worth notice in a tentative way.

Suppose that M is the ensemble of elements in the continuum. Let their number be $\aleph_1$.* The set $\mathbf{U}(M)$ according to those who believe in such things would have a greater number, say $\aleph_2$. The very introduction of this set into the reasoning on the well-ordering of the continuum implies the use of something which is more transfinite than is necessarily called for by the theorem to be proved. And this greater transfinitude must be in working order if Zermelo's correspondence is to be obtainable. What might be the order of infinitude which shall suffice for specifying the number (a) of ways in which the correspondence may be set up is a question which will be left to the reader. There are, however, indications which point with considerable assurance to the inference that if Zermelo's reasoning were followed to its (logical?) conclusion, it would result in establishing the fact that the continuum may be well ordered in something like $\aleph_1$! ways. If so, is the theorem worth having? Does it mean anything?

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*I do not wish to imply the existence of the transfinite cardinals: the three aleph symbols here introduced may be regarded as notations for hypothetically existing entities.

To get a glimpse of the bearing of these questions it may be well to look at the problem of well-ordering the positive rational numbers. The method of ordering them by telling them off against the positive integers is well known and is valuable. Suppose Zermelo's method were followed. Here $\mathfrak{U}(M)$ would be that set which is introduced in defining the continuum — a set of greater infinitude than M. Suppose next that the correspondence required by the author were set up, the set $\mathbf{M}$ picked out, and the ordering accomplished. In view of the evidence that this could all be accomplished (if at all) in a high infinitude of ways, is it evident that any more (and perhaps even less) has been accomplished than would have been accomplished by the simple statement that to order and well-order the positive rational numbers it is merely necessary to pick out one of them and set it down and then pick out another and set that down next to the former, and so on?*

To sum up this discussion I should say:

1° that Zermelo is right in his contention that he may add his postulate to the existing logical system; that to deny him that privilege would be to put an embargo on the development of logic and to assume a completeness of our logical system which is quite unwarranted in view of past developments and future possibilities:

2° that the use he makes of his principle is to render a hard thing harder, a transfinite condition more transfinite; that there still remains so much of an arbitrary nature in the determination of the correspondence as to make his theorem on well-ordering of little more value than the statement that the elements of any set may be plucked out seriatim and set in a row; that he applies this latter principle (except for the seriatim plucking) to a case which really transcends the original set in infinitude:

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* This idea has been stated by Borel, "Quelques remarques sur les principes de la théorie des ensembles," *Mathematische Annalen*, vol. 60, pp. 194–195. For comment, see the next footnote.

3° That the well-ordering of any set is of practically no significance and is quite worthless apart from an algorithm which accomplishes the ordering — an algorithm which shall not require an operation which transcends the cardinal number of the given set.*

It should be added that, entirely apart from the problem of well-ordering, the application of Zermelo's principle to the construction of a general theory of sets (Mengenlehre) along the lines he is now following is an interesting study which may contribute in no small measure toward the solution of some of the fundamental questions of logic and may easily result in the general adoption of some form of the principle which he has introduced explicitly and which has tacitly been used to a very considerable extent with no overwhelmingly fallacious results. It seems far from certain that we shall never need a more general principle of reasoning than that contained in the present acceptation of the scope of mathematical induction.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, MASS., *April*, 1908.

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* In forming any conclusions concerning Zermelo's problem one must not overlook the keen remarks made on both sides of the question by several authors and published under the title "Cinq lettres sur la théorie des ensembles" in the *Bulletin de la Société Mathématique de France*, vol. 33, pp. 261–273. My conclusions here stated agree very largely with those of Borel and Baire in those lettres; and hence there would be little need of my making the restatement that I do, were it not for the fact that I feel that I have treated the matter somewhat more in detail and in a way less open to objection on the ground of subjectivity or prejudice. Hadamard pointed out in his first letter (and this has an application to the last statement in 2°) that there was a difference between picking out one element after another from an infinite set and setting them in order on the one hand and selecting an element from each one of an infinity of independent sets on the other hand. If that be granted (it is not granted in the present case by all who have commented on the subject), I still fail to see how there is any great difference between plucking one element after another from the continuum and selecting an element from each one of the subsets of the continuum, especially in view of the fact that the set of subsets is of greater number (Mächtigkeit) than the continuum. It is the application of the principle of selection to this set of greater number which particularly impresses me; and in addition to this, the so great freedom of choice which is to be found in the selection. The freedom appears quite sufficient to enable the well-ordering to be likened to plucking one element after another. So far as I am aware, these two fundamental considerations have not been taken up by previous writers. This, with the discussion of logic which occupies the first half of the paper, may perhaps be a sufficient justification for my present communication.

Postscript. — Since the above was sent to the printer, Schoenflies's report on Die Entwickelung der Lehre von den Punktmannigfaltigkeiten, 2ter Teil, has come to hand and should be included among the references cited in this paper. On page 36, paragraph 3, the author states a conclusion which is practically that of Borel in the *Annalen*, loc. cit., volume 60, namely, that Zermelo's proof can be characterized as establishing the equivalence of the principle of selection and the principle of well-ordering. That is to say, if a given set is well-ordered a correspondence such as Zermelo postulates may be set up between the subsets and the elements of the set by simply associating to each subset its first element, and Zermelo's proof shows that conversely if the principle of selection is admitted the given set may be well-ordered.

I am by no means sure that these statements bring the two principles into equivalence, as Borel and Schoenflies assume. It should be remembered that if a set is well-ordered we can actually associate to *any* subset an ausgezeichnetes element, whereas the postulate of selection requires us by an arbitrary process to associate to *all* the subsets an element. As far as finite sets go, the distinction between any and all is not so great as not to be readily overlooked, although logicians regard a postulate concerning any element of a set as far simpler than one concerning all elements. When, however, it is a question of infinite sets and particularly of non-enumerable sets the difference between any and all is vast, and many statements or proofs, unexceptionable when phrased in terms of any, are by no means acceptable when all replaces any.

In view of these considerations and the oft-repeated fact that Zermelo's principle must be applied to a set of greater number than the one to be ordered, I should seriously doubt the equivalence between his principle and that of ordering and should feel quite justified in claiming that it was logically and mathematically preferable to take the principle of ordering as a postulate instead of the principle of selection in the general form in which Zermelo states it.

E. B. W.

May, 1908.

SHORTER NOTICES.

Sur le Développement de l'Analyse et ses Rapports avec diverses Sciences: Conférences faites en Amérique. Par EMILE PICARD. Paris, Gauthier-Villars, 1905. 167 pp.

La Science moderne et son Etat actuel. Par EMILE PICARD. Paris, E. Flammarion, 1908. 301 pp.

PICARD'S lectures delivered here in America, three at the decennial of Clark University in 1899 and one at Saint Louis in 1904, were published soon after their delivery. Those in celebration of the decennial of Clark appeared in the volume devoted to that celebration and again in the *Revue générale des sciences* early in 1900. The lecture at Saint Louis had its place in the Proceedings of the congress of arts and sciences. It was not, however, until 1905 that the author collected the four lectures in a separate monograph readily accessible to all.

In the general introduction to the first lecture the author states: "Je lis toujours pour ma part . . . le *Bulletin* de la Société mathématique américaine . . . qui tient ses lecteurs au courant des travaux les plus récents." It is much to be feared that the present review is too tardy to come under this generous compliment. There is, however, in all that Picard writes, and particularly in these lectures, that finish and that wisdom which makes his work immediately a classic and compels, not at the time merely but for many years thereafter, the attention and interest of the student who would really be "au courant des travaux les plus récents."

The chief thoughts in the first lecture are grouped about the genesis and growth of the general conception of function. The subject takes one far and wide in the history of the development of mathematics and there is scarcely a reputable mathematician since Euler whose name does not deserve mention in the sketch; for although the functions of real and of ordinary complex variables naturally take most of the attention, the possibility of functions of other hypercomplex numbers and the various geometric fields therewith connected are not overlooked. That the author is not so exclusively concerned with the scientific side of his subject as to forget the fundamental pedagogic ideas, is brought into evidence by the two remarks: "Si Newton et Leibniz avaient pensé que les fonctions continues n'ont pas nécessairement une dérivée, ce qui est le cas général, le calcul différentiel n'aurait pas pris naissance; . . . " and in reference to the extremes to which rigor may be pushed "ces spéculations raffinées ont même pénétré dans l'enseignement élémentaire, ce qui est, a mon avis, très regrettable." And such matters are not merely mentioned; they are treated as worthy of some elaboration.

The second lecture is on the theory of differential equations, whether partial or ordinary. Many interesting results are cited from the work of the previous decade, and not a few of them are due to the author or to others who have followed where he has led. One fact that is rarely seen in print is the discovery of Borel that every analytic integral of a partial differential equation with analytic coefficients may be expressed by a formula containing only one arbitrary function of a single real variable. In regard to ordinary equations, the work of Painlevé on equations which are not linear and of Poincaré on the configuration of the real solutions are among the foremost of the matters noticed. The author very naturally records a protest against the tendency to regard the theory of ordinary equations as a chapter in the theory of analytic functions.

That Picard would hardly subscribe to the motto "Es giebt nur Potenzreihen" appears evident from the fact that much of the discussion in the first two lectures is connected with the name of Fourier and that it is only in the third lecture that he comes to speak especially of analytic functions and of the various special functions which have been introduced into mathematics. After a preliminary reference to the origin of the theory of analytic functions and an extended mention of the recent work of the French school, among whom Mittag-Leffler must be included, on the various developments of these functions and on the special properties of integral functions, the author draws the general outlines of the automorphic functions, the new transcendents of Painlevé, the hyperautomorphic functions, and finally his own algebraic functions of two variables. Desiderata are not forgotten amid the mass of results already acquired.

The concluding lecture, that at Saint Louis, wherein the general development of mathematics in connection with its applications is reviewed from early times through the century of d'Alembert's bon mot "allez en avant, et la foi vous viendra" down to the present rigorous days, cannot be sub-reviewed: it must be read. One idea, however, which will appear again in

La Science moderne, may be noted: that of heredity as against non-heredity, of functional equations as against differential equations.

Modern science as it is is a subject so vast that no one person can hope to be an "Ausgelernter" in science alone, to say nothing of all other domains of thought, as might have been possible in the days of Faust. Yet if there is a place where such an accomplishment is most favored it is Paris, where the five academies flourish and where intellectual pursuits are centralized as they are nowhere else. Picard's book, La Science moderne, may well take its place among the fair flowers of this hothouse of thought.

In the introduction, after shutting out metaphysical considerations and limiting himself to a discussion of the methods of investigation which are essentially scientific, the author points out the constantly growing field of science and the constantly increasing correlation of different parts of the field, until "Il semble que dorénavant, dans la vie scientifique comme dans la vie sociale, l'association s'imposera de plus en plus. Tel travail ne pourra être effectué que par la collaboration d'un mathématicien et d'un physicien, et tel autre demandera le concours d'un chimiste et d'un physiologiste." This is a generalization and at the same time a reminiscence of the idea with which he closed his lectures at Clark—an idea which in practice is constantly and emphatically outraged by the training that is provided by some of our foremost schools as preparation for the doctorate—"En terminant, je me permettrai de donner un conseil aux étudiants mathématiciens qui m'ont fait l'honneur de m'écouter: je leur recommanderai de ne pas se cantonner trop tot dans des recherches spéciales. Il leur faut acquérir d'abord des vues générales sur les diverses parties de notre science, sans lesquelles leurs recherches risqueraient de rester stériles, et qui leur coûteraient plus tard un bien plus grand effort."

The first chapter is a summary of the lecture at St. Louis, explained so far as may be without technicalities and adapted to the comprehension of a more popular audience. The second chapter bears the title, the mathematical sciences and astronomy. There is much more in it than the title would indicate; for a start is made from the modern theories of number and of analysis and from some rapid indications concerning recent advances in geometry, with especial reference to the theory of groups (it should be remembered that the new program of public instruction in France introduces the notion of groups into geometry at a very early stage) and of the popularly attractive non-euclidean geometries. And some further words on the development of analysis are inserted before the author finally settles down to celestial mechanics and astronomical physics. So much for the venerable sciences of mathematics and astronomy.

The next chapter logically comes to the subjects of mechanics and energetics. Here there is an account of mechanics from its early formative stages through the time of Newton ; some mention of the deductive side of mechanics from that day to the present, with especial emphasis on the ideas of Hertz in regard to hidden motions and hidden masses ; and then the statement and discussion of the problem of the mechanical explanation of the universe. There is a concluding section on energy, in which the laws of thermodynamics and the relation of mechanics to the wider fields of physics receive attention. The fourth chapter is on the physics of the ether. The way in which the author manages to set forth the interplay of optics with electricity and magnetism, to tell of the recent discoveries in cathode and X-rays, and to expound the chief phenomena and theories of the radioactive substances is to be admired and beheld with wonder.

That Picard is proceeding from the more abstract and definite toward the more phenomenological and less thoroughly correlated sciences may be seen from the titles of the remaining five chapters : material physics and chemistry, mineralogy and geology, physiology and biological chemistry, botany and zoology, and medicine and the microbic theories. Whether or not the first four chapters would be readily understood by persons who, though educated, were not specialists in mathematics or physics might at first be a serious question : but if the explanation there is as clear and interesting as it surely is in these five last chapters, which are certainly not directly in our province, the answer could only be in the affirmative. To read, in the author's delightful French, of the accomplishments in all these fields by such a variety of authorities as Moissan, Suess, Koch, Lehmann, Haeckel, Pasteur, Loeb, Marsh, de Vries, and many others whose names are less familiar, is highly interesting and leaves the same literary and romantic impression as Maeterlinck's L'Intelligence des fleurs, less the fantasy — but would scarcely find a proper place for review in this periodical. Besides, it should appear that even if one man did write of all these matters, one reviewer could not be expected to treat all of them in other than the perfunctory way in which almost all offerings of scientific work are "reviewed" in the daily or weekly press.

E. B. WILSON.

Algebraic Equations. By G. B. MATHEWS, M.A., F.R.S., Fellow of St. John's College. Cambridge University Press, 1907. 64 pp.

THIS is one of a half dozen "Cambridge Tracts in Mathematics and Mathematical Physics," issued by the Cambridge University Press. In this tract the author is confronted with the problem of giving an exposition of the Galois theory of equations within the compass of about sixty pages. This limitation makes it necessary for him to present the subject more or less in outline and to confine himself to a very few illustrative examples. An outline presentation, prepared by an eminent author, is certain to bring out in bold relief interesting view points. Such is the case in this booklet. And yet we are of the opinion that the real value of this book to beginners would have been enhanced by more abundant illustration and a somewhat fuller detail of explanation.

To save space, the author does not put down a definition in a sentence by itself; the definition is to be inferred from a condensed statement made as part of a sentence occurring perhaps in the body of a demonstration. Thus, the definition of an intransitive group (page 14) is given in course of a proof, as follows: "First suppose G is intransitive: this means that a certain number of roots $x_1, x_2, \cdots, x_r$ $(r < n)$ are only interchanged among themselves by the substitutions of G." Less easily comprehended are the definitions of simple groups and self-conjugate factor groups, similarly interpolated on pages 16 and 17. Despite the effort to secure extreme condensation, there occur redundancies, such as "absolutely undetermined" (page 2), "absolutely unaltered" (page 57), "perfectly definite" (page 6). These are less objectionable in oral exposition than in a printed outline.

Here and there are evidences of hasty composition. Thus, the tract is encumbered with some heterogeneous terminology. The author speaks in different places of the "arithmetically rational," and the "rational in the ordinary sense," also of a "definite numerical irrationality" and an "irrational quantity." The terms "rational" and "rational function" are not always accompanied with an explicit statement of the field of rationality, so that extra care must be exercised by the reader, lest he misinterpret the author. On page 5 is proved the theorem that any rational function of the roots of a given equation $f(x) = 0$ can be expressed as a rational function of any one of the roots of its complete galoisian resolvent. This is done at a time when the meaning of the term "rational" has not, as yet, been extended to apply to different fields. However, the generalization is made on the next page. As a corollary to this theorem, it is shown that all the roots of a complete galoisian resolvent may be expressed as rational functions of any one of them; an equation having this property is called a normal equation. As this complete galoisian resolvent may be reducible in the field defined by the coefficients of the given equation, it would follow that a normal equation may be reducible — a definition that is both undesirable and contrary to the usage of writers on the Galois theory.

An unfortunate phraseology occurs on pages 2, 3 and 4, in connection with "arbitrary" and "numerical" coefficients. It leaves the impression that equations of higher degrees than the fourth are algebraically unsolvable only when the coefficients are "arbitrary." "If $n < 5$ and the coefficients are left arbitrary, it is possible to construct an explicit algebraic function of the coefficients which is a root of the equation. For $n > 4$, this is no longer the case." As if this statement did not apply at all to equations with numerical coefficients, the author proceeds to say: "When the coefficients are numerically given," the roots can be found by trial or approximation, although the main problem with him is this: "Given a particular equation with numerical coefficients, it is required to find the simplest set of irrational quantities such that all the roots of the given equation can be expressed as finite rational functions, in an explicit form, of the set of irrationals." Returning to arbitrary coefficients, he gives then a preliminary discussion of the cubic, quartic and quintic. Nowhere on these pages does he convey the idea that numerical equations may be algebraically unsolvable, nor does he here make plain that, besides the extreme cases of coefficients that are all "arbitrary" or all "numerical," there are intermediate assumptions which demand our attention. Needless to say, no one knows the facts in question better than our distinguished author, and it is a pity that he did not take time to express himself more precisely. Later the author drops the term "arbitrary coefficients" and speaks of "coefficients represented symbolically" (page 11).

The foundation of the Galois theory covers in this tract the first twenty-nine pages; then eleven pages are given to cyclic equations, four pages to abelian equations and ten pages to metacyclic equations. The last chapter is upon "Solution by standard forms" and contains, on six pages, in outline, the treatment of the quintic by means of the icosahedral irrationality. This outline can be used with interest in conjunction with Klein's Ikosaeder.

FLORIAN CAJORI.

Introduzione alla Geometria Proiettiva degli Iperspazi. By E. BERTINI. Pisa, Spoerri, 1907. 425 pp.

THE subject of geometry of n dimensions has undergone a noteworthy advancement during the last few decades. In this advancement the Italian geometers have taken a very prominent part. It is therefore fitting that this able work on projective geometry of n dimensions should be the work of an Italian.

The author avoids the difficulties of spacial intuition by starting from a purely arithmetical definition of a point and of a space. There exist, in ordinary geometry, numerous examples of systems of entities which can be put into one to one algebraic correspondence, without exceptional elements, with the values of the ratios of a system of parameters. The properties of such systems are of two kinds, those depending on the particular nature of the entities and those depending only upon the fact of the correspondence with the group of values of the parameters. The latter properties the author considers to form the subject matter of projective geometry of n dimensions. A point is therefore defined as a set of ratios of $n + 1$ parameters $x_0, x_1, \cdots, x_n$, and a space of n dimensions as "the totality formed by all the sets of values (real and complex) of these ratios."

The first two chapters are devoted to a discussion on this basis of the ideas of subordinate spaces, section, projection, etc. The possibility of a satisfactory synthetic treatment is also pointed out, however, and a number of references to such treatments are given.

The next three chapters, of about eighty-four pages, are devoted to the discussion of projective and dualistic correspondences. Chapters six and seven, containing each about twenty pages, treat of the properties of quadrics and systems of quadrics. Chapter eight, of twenty-five pages, is devoted to the discussion of hypersurfaces and chapter nine, of thirty pages, to the discussion of algebraic spreads in general. In chapters ten and eleven, containing about fifty pages, an able and interesting discussion of linear systems of hypersurfaces is given. Chapter twelve, of fifteen pages, treat of rational curves, with especial attention to rational normal curves. Chapter thirteen, containing twenty-three pages, treats of rational ruled surfaces. In chapter fourteen, of about twenty pages, the author discusses the relations of linear systems of plane curves to two dimensional spreads in space of n dimensions. Chapter fifteen, of twenty-five pages, is devoted to a thorough discussion of the surface of Veronese and its three dimensional projection, the Steiner surface.

Of especial importance is the appendix of about fifty pages. This is divided into three chapters. The first deals with branches of algebraic curves, the second with quadratic transformations and the third with correspondences on one dimensional spreads.

It will be seen that the book covers only a portion of the field of geometry of n dimensions; but the subjects treated are discussed with a great deal of thoroughness. The treatment is in fact, in some cases, rather profuse. It would, perhaps, have been better to omit some of the minor discussions except for a reference to where the treatment may be found.

A regretable feature of the book, from the standpoint of the general reader, is the scarcity of references. This lack is recognized by the author himself, but he points out that the book is intended primarily for undergraduates in the Italian universities, for whom an extensive bibliography did not seem so necessary.

The reader will find in no other place so complete and comprehensive an exposition of our present knowledge of projective geometry of n dimensions as is given in this book. It is clearly and logically written and unusually free from errors. It will doubtless stimulate a much wider interest in this field of research.

C. H. SISAM.

Analytische Geometrie des Punktes, der geraden Linie und der Ebene. Von OTTO STAUDE. Leipzig, B. G. Teubner, 1905. viii + 447 pp.

THIS volume is offered as an introduction to the theory of surfaces of the second order, which the publishers requested Professor Staude to write in connection with the Encyklopädie der mathematischen Wissenschaften. It is largely concerned with the various systems of coordinates used in the study of points, lines, and planes. Each subject is based upon cartesian coordinates, from which the other systems are derived.

The text is divided into three parts: the first treats of ranges of points and plane pencils; the second deals with the point, line, triangle, and quadrangle; the third discusses configurations of lines and planes. The subject matter is not new; the method is elegant.

Two characteristic features of this excellent work are the thoroughness with which the fundamental notions have been discussed, and the large place given to the principle of duality. The historical and critical notes of the appendix are exceedingly valuable. The author indicates the important contributions of various geometers, states the sources of new conceptions, indicates parallel lines of development, and gives a multitude of references.

G. N. BAUER.

Problems in Strength of Materials. By WILLIAM KENT SHEPARD. Ginn and Co. viii + 72 pp.

THE purpose of this book, another of the Yale series of mathematical texts, is to furnish a large variety of problems to supplement and illustrate the theory of the strength of materials. The author believes that, in order to obtain a working knowledge of any scientific subject, the average student should solve numerous problems illustrating the application of the theory, and that the present text-books on the strength of materials do not furnish a sufficient number.

The 568 problems proposed cover fully the field of the subject, as usually taught in scientific and technical schools. The fundamental subjects of tension, compression, shear, and elastic deformation are first considered; after which the problems are mainly those of the design and investigation of cylinders and spheres, riveted joints, the various types of beams and columns, plates, and shapes to resist torsion and combined stresses.

The use of the factor of safety in designing is emphasized throughout.

The purpose of the author evidently is to make the problems as practical as possible; this he might have realized more fully if, for instance in the design of columns, he had given specific illustrations of such shapes actually designed and erected. Again, while the figures throughout the book are excellent and make their point, they would have been more efficient if, for example, in the case of riveted joints, they had represented actual sections, correct from the standpoint of engineering practice.

The typography of the book is up to the high standard of all of the texts of the Yale series. Tables of constants and data usually included in engineers' handbooks are given at the back.

ERNEST W. PONZER.

Das 200-jährige Jubiläum der Dampfmaschine, 1706–1906. Eine historisch-technisch-wirtschaftliche Betrachtung. Von KURT HERING, Ingenieur. Leipzig, Berlin, B. G. Teubner, 1907. Pp. 58.

THIS pamphlet is not put forth as a history of the steam-engine. Such a history could not very well be written within so small a compass. Prepared at the 200th anniversary of Denis Papin's invention, as set forth in his "Ars nova" (1707), the pamphlet devotes itself mainly to an exposition of the ideas and experiments of this noted French physicist, who for many years resided in Germany. Papin is looked upon by the author as the theoretical inventor ("geistige Erfinder") of the steam engine. It is unfortunate that an exact date (1706) is set up in the title of the publication for the invention. This date fixes Papin as the inventor, but such a claim the author does not really make in the body of the book. We translate from his introduction: ". . . but who was the inventor of this most important machine of modern times? Not as the work of one only, but as the product of many scholars and practical men of the most different nations, does the steam engine come down to us." One historical point, discussed by Hering, needs emphasis here, namely that Papin never constructed a steamboat and that the boat on which he travelled on the river Fulda from Kassel to Münden on September 24, 1707, though embodying the novel idea of paddle wheels, was not driven by steam. Since 1690 Papin had been considering the idea of applying steam power to boats, but in the description of a boat under construction, given in a letter to Leibniz (March 13, 1704), Papin says: "Je n'ay point préparé celui ci pour y emploier la force du feu: parceque ce n'est pas à moi d'entreprendre trop des choses à la fois." In Robert H. Thurston's History of the growth of the steam engine (New York, 1893), we read (pages 224, 225) that Papin propelled his boat by his "steam engine" and that "an account of his experiments is to be found in manuscript in the correspondence between Leibniz and Papin, preserved in the Royal Library at Hanover." As Thurston does not give the date of the letter, nor quote from it, while Hering supports his contention also by quoting E. Gerland, who for thirty years has been engaged in editing the correspondence between Leibniz, Papin, and Huygens and the Leibnizian manuscripts on mechanics, and who stoutly combats the conclusion that Papin ever built himself a boat driven by steam, it would seem that Thurston must be in error. Were there any real support to the story of Papin's steamboat, then surely the Germans would be the last to deny it.

The pamphlet closes with a chapter embodying statistics and charts to show the effect of the introduction of the steam engine upon industrial life, particularly in Germany.

FLORIAN CAJORI.

NOTES.

AT the April meeting of the Council of the AMERICAN MATHEMATICAL SOCIETY, Professor E. B. VAN VLECK was re-elected a member of the Editorial Committee of the *Transactions*.

THE April number (volume 9, number 3) of the *Annals of Mathematics* contains the following papers: "A necessary condition that all the roots of an algebraic equation be real," by O. D. KELLOGG; "The equilibrium of a heavy homogeneous chain in a uniformly rotating plane," by E. B. WILSON; "The continuity of the roots of an algebraic equation," by J. L. COOLIDGE; "On the differentiation of definite integrals," by W. F. OSGOOD; "Note on the convergence of a sequence of functions of a certain type," by H. S. BUCHANAN and T. H. HILDEBRANDT; "On the inverse problem of the calculus of variations," by G. A. BLISS; "A geometrical interpretation of the generalized law of the mean," by C. N. HASKINS; "On the torsion of a curve," by P. L. SAUREL.

AT the meeting of the London mathematical society held on April 30 the following papers were read: By E. W. HOBSON, "On a general convergence theorem and the theory of the representation of a function by series of normal functions"; by G. H. HARDY, "On the multiplication of series"; by F. H. JACKSON, "On q integration and differential equations."

THE Guccia medal was awarded Monday, April 6, at the first session of the fourth international congress of mathematicians. No one of the competing memoirs was deemed worthy of the prize, and according to the conditions (BULLETIN, volume 11, page 167) it was to be given to the author of the most important memoirs in algebraic geometry during the interval from November, 1904, to July, 1907. The medal was consequently unanimously awarded to FRANCESCO SEVERI, professor of mathematics at the University of Padua.

AT the Rome meeting of the fourth international congress of mathematicians a committee of organization was appointed to investigate the teaching of mathematics in various countries and report at the next congress in 1912. This committee, consisting of Professor F. KLEIN, of the University of Göttingen, Professor A. G. GREENHILL, of the Ordinance College, Woolwich and Professor H. FEHR, of the University of Geneva, was charged with the duty of selecting the members of the general committee after consulting with the councils of the leading mathematical societies of the various countries.

THE Deutsche Mathematiker Vereinigung will hold its annual meeting in affiliation with the eightieth convention of the society of German naturalists and physicians at Cologne during the week beginning September 20. While papers on mechanics will form the chief subject for discussion, memoirs on other subjects are welcome, and should be reported to the secretary, Professor A. KRAZER, of Karlsruhe, at once. Professor F. KLEIN, of Göttingen, is president during the present year.

THE fourth volume (1759–1799) of Cantor's Geschichte der Mathematik has recently appeared, the first copy having been laid before the international congress at Rome.

THE complete report of the investigation made by the *Enseignement Mathématique* and the department of psychology of the University of Geneva regarding the methods of work of various mathematicians has been republished in a pamphlet of 126 pages by the firm of Gauthier-Villars in Paris under the title Enquête de l'Enseignement Mathématique sur la méthode de travail des mathématiciens.

FROM a pamphlet recently issued by the departments of mathematics and mathematical astronomy of the University of Chicago it appears that from the opening of the University in 1892 to October, 1907, 1194 students pursued in these departments studies above and including the calculus. Of these 689 were registered in the graduate school. The number of major courses offered from the calculus upward was 624. The master's degree was conferred on 50 candidates, the doctor's degree on 33.

A NUMBER of mathematical books from the library of the late Professor Heinrich Maschke are offered for sale at advantageous prices. A list of the books can be obtained from Professor Oskar Bolza, University of Chicago, to whom correspondence should be addressed.

DR. A. TAUBER, of the University of Vienna, has been promoted to an assistant professorship of mathematics.

DR. R. ROTHE, of the technical school at Charlottenburg, has been appointed professor of mathematics at the academy of mines at Clausthal.

PROFESSOR J. WEINGARTEN, of the University of Freiburg, has been declared professor emeritus, with the rank of full professor.

PROFESSOR M. CANTOR, of the University of Heidelberg, has retired from active teaching with the title of honorary professor.

PROFESSOR K. HENSEL, of the University of Marburg, has been elected a member of the academy of sciences of Halle.

PROFESSORS JOSEPH LARMOR and W. E. STORY have been elected members of the National Academy of Sciences.

PROFESSOR MAX MASON, of Yale University, has been appointed associate professor of mathematics at the University of Wisconsin.

At the University of Michigan, Mr. W. H. Butts has been appointed to a junior professorship of mathematics.

At the Michigan Agricultural College Dr. Carl Gundersen has been promoted to an assistant professorship of mathematics.

Dr. A. G. Hall, of Miami University, has been appointed to a professorship of mathematics at the University of Michigan.

Mr. H. F. Hart has resigned his instructorship in Syracuse University to accept the position of teacher of mathematics in the Montclair, New Jersey, high school.

Professor Peter Field, of the University of Michigan, has been granted a leave of absence for next year, to study in Europe.

Professor G. N. Bauer, of the University of Minnesota, has received leave of absence until autumn of 1909, and will study at the University of Göttingen.

Dr. G. Picciati, who had just been appointed professor of theoretical mechanics at the University of Bologna, died March 11 at the age of 39 years.

Professor S. M. Kepinski, of the technical school at Lemberg, died March 28 at the age of 40 years.

Professor W. Scheibner, of the University of Leipzig, died April 8, at the age of 82 years.

Professor A. Mayer of the University of Leipzig, editor of the *Mathematische Annalen,* died April 11, at the age of 69 years.

Recent catalogues of second-hand mathematical books: Henry Sotheran and Co., 140 Strand, London, catalogue No. 682, about 4,600 titles in mathematics and sciences. — C. Lang and Co., Via Bocca di Leone, 13, Rome, catalogue No. 3, about 800 titles in general science. — Buchhandlung Gustav Fock, Schlossgasse 7–9, Leipzig, catalogue No. 331, journals and collections, mathematics, physics, and astronomy, 22 pp.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Bachl (F.). Ueber dreifach orthogonale Flächensysteme. (Progr.) Wien, 1907. 8vo. 13 pp.

Bonola (R.). Die nichteuklidische Geometrie. Historisch-kritische Darstellung ihrer Entwickelung. Autorisierte deutsche Ausgabe besorgt von H. Liebmann. Leipzig, Teubner, 1908. M. 5.00

Börner (E.). Ueber irreducible Nichtquaternionen-Systeme mit sechs Haupteinheiten. (Progr.) Mährisch-Ostrau, 1907. 8vo. 44 pp.

Bromwich (T. J. I'A.). An introduction to the theory of infinite series. London and New York, Macmillan, 1908. 8vo. 13 + 511 pp. Cloth. $4.50

Burkhardt (H.). Funktionentheoretische Vorlesungen. Vol. I. 2tes Heft: Einführung in die Theorie der analytischen Funktionen einer komplexen Veränderlichen. 3te, durchgesehene und vermehrte Auflage. Leipzig, Veit, 1908. 8vo. 12 + 262 pp. Cloth. M. 8.00

Cairns (W.). Die Anwendung der Integralgleichungen auf die zweite Variation bei isoperimetrischen Problemen. (Diss.) Göttingen, 1907. 8vo. 68 pp.

Cantor (M.). Vorlesungen über Geschichte der Mathematik. Vol. IV. Von 1759 bis 1799. 4te Lieferung. Abschnitt XXVI: G. Vivanti, Infinitesimalrechnung. Leipzig, Teubner, 1908. 8vo. Pp. 643-882. M. 7.00

Doehlemann (K.). Geometrische Transformationen. 2ter Teil: Die quadratischen und höheren, birationalen Punkttransformationen (Sammlung Schubert, XXVIII). Leipzig, Göschen, 1908. 12mo. 8 + 328 pp. Cloth. M. 10.00

Durège (H.). Theorie der elliptischen Funktionen. 5te Auflage, neu bearbeitet von L. Maurer. Leipzig, Teubner, 1908. 8vo. 8 + 436 pp. Cloth. M. 11.00

Garbieri (G.). Geometria analitica: riassunto di lezioni date nell'università di Genova. Parte II (Luoghi di 2° grado in coordinate cartesiane con numerosi esercizi). Torino, Paravia, 1908. 8vo. 76 pp. L. 4.00

Günther (S.). Geschichte der Mathematik. 1er Teil: Von den ältesten Zeiten bis Cartesius (Sammlung Schubert, XVIII). Leipzig, Göschen, 1908. 12mo. 7 + 427 pp. Cloth. M. 9.60

Hartl (H.). Erste Einführung in die Elemente der Differential- und Integralrechnung und deren Anwendung zur Lösung praktischer Aufgaben. Wien, Deuticke, 1908. 8vo. 3 + 59 pp. M. 1.20

Janzen (O.). Ueber einige stetige Kurven, über Bogenlänge, linearen Inhalt und Flächeninhalt. (Diss.) Königsberg, 1907. 8vo. 70 pp.

Khandrikov (M.). Infinitesimal Analysis. Vol. III. (Russian.) St. Petersburg, 1907. 8vo. M. 12.00

Kischke (R.). Ueber Fehlerabschätzung bei unendlichen Produkten und deren Anwendungen. (Diss.) Königsberg, 1908. 8vo. 83 pp.

KÜBEL (K. G. J.). Anwendungen einer anschaulichen Darstellung des Imaginären. (Diss.) Giessen, 1907. 8vo. 27 pp.

LALESCO (T.). Sur l'équation de Volterra. (Thèse.) Paris, Gauthier-Villars, 1908. 4to. 392 pp.

MAURER (L.). See DURÈGE (H.).

SCHEIBNER (W.). Beiträge zur Theorie der linearen Transformationen, als Einleitung in die algebraische Invariantentheorie. Leipzig, Teubner, 1907. 8vo. 250 pp. M. 10.00

SCHUBERT (H.). Niedere Analysis. 1er Teil: Kombinatorik, Wahrscheinlichkeitsrechnung, Kettenbrüche und diophantische Gleichungen. 2te Auflage. (Sammlung Schubert, V.) Leipzig, Göschen, 1908. 12mo. 4 + 181 pp. Cloth. M. 3.60

VIVANTI (G.). See CANTOR (M.).

II. ELEMENTARY MATHEMATICS.

ARBES (J.). Ueber Erfindung, Gestaltung und Wertschätzung der Logarithmen. (I.) (Progr.) Smichow, 1907. 8vo. 22 pp.

BARINGER (W.). Was muss man von der Arithmetik und der Algebra wissen? 2te Auflage. Berlin, Steinitz, 1907. 8vo. 71 pp. M. 1.00

BARONI (E.). Algebra e trigonometria, ad uso dei licei, secondo gli ultimi programmi. Vol. I (Primo anno di liceo). Firenze, Bemporad, 1908. 16mo. 6 + 363 pp. L. 2.75

BORTOLOTTI (E.). Nozioni pratiche di geometria, per le scuole complementari. 3a edizione, accuratamente riveduta. Roma, 1908. 8vo. 174 pp. L. 1.75

BOURDON. Eléments d'algèbre. 20e édition, revue et annotée par E. Prouhet. Paris, Gauthier-Villars, 1907. 8vo. 12 + 657 pp. Fr. 8.00

BRUNO (G. M.). Curso superior con numerosos ejercicios y nociones de agrimensura, levantamiento de planos y nivelación. Paris, 1908. 16mo. 400 pp.

CHAILAN (E.). Géométrie à l'usage des élèves de l'enseignement secondaire. Premier cycle, divisions A et B; second cycle, divisions A et B (programme de 1905). 4e édition. Paris, Poussielgue, 1908. 16mo. 8 + 240 pp.

DREYFUS (L.). See FORT (L.).

FENKNER (H.). Lehrbuch der Geometrie für den Unterricht an höheren Lehranstalten. (In 3 Teilen.) 3er Teil: Ebene Trigonometrie. Nebst einer Aufgabensammlung. Berlin, Salle, 1908. 8vo. 4+102 pp. M. 1.60

FORT (L.) et DREYFUS (L.). Eléments de géométrie, conformes aux programmes du 27 juillet, 1905. Classes de cinquième B et de quatrième A, avec de nombreux exercices, figures et problèmes. 2e édition. Paris, Paulin, 1908. 18mo. 8+215 pp. Fr. 2.50

HÄRING (G.). Lehrbuch der Geometrie für die Oberstufe der höheren Lehranstalten und zum Selbstunterricht. München, Oldenbourg, 1908. 8vo. 5+96 pp. M. 1.30

LOEWENBERG (G.). Was muss man von der sphärischen Trigonometrie und der Astronomie wissen? Berlin, Steinitz, 1907. 8vo. 77 pp. M. 1.00

LUDWIG (F. K.). Das logarithmische Rechnen. Leichtfassliche Darlegungen über das Wesen, die Berechnung und Anwendung der Logarithmen nebst zahlreichen Beispielen mit ausführlichen Lösungen. Zum Selbstunterricht bearbeitet. 2te Auflage. Komotau, Benker, 1908. 8vo. 53 pp. M. 1.30

MEHLER (F. G.). Hauptsätze der Elementar-Mathematik zum Gebrauche an höheren Lehranstalten. Neubearbeitet von A. Schulte-Tigges. Ausgabe A. 25ste Auflage des Stammbuches. Berlin, Reimer, 1908. 8vo. 12 + 280 pp. M. 2.40

PROUHET (E.). See BOURDON.

SCHULTE-TIGGES (A.). See MEHLER (F. G.).

III. APPLIED MATHEMATICS.

BOLTZMANN (L.). Vorlesungen über Maxwell's Theorie der Elektricität und des Lichtes. 2 Teile. 2ter unveränderter Abdruck. Leipzig, Barth, 1908. 8vo. M. 6.00

CAMPBELL (N. R.). Modern electrical theory. Cambridge, University Press, 1908. 8vo. 232 pp. Cloth. $2.25

D'OCAGNE (M.). See LECORNU (L.).

ENACHE (M.). Hydrodynamique. Contribution à la théorie de l'écoulement sur les déversoirs en mince paroi et à nappe voyée en dessous. (Thèse.) Paris, Gauthier-Villars, 1908. 4to. 37 pp.

JEANS (J. H.). The mathematical theory of electricity and magnetism. Cambridge, University Press, 1908. 8vo. 8 + 536 pp. Cloth. $4.50

JORDAN (H.). Zur Kenntnis der elastischen Nachwirkung bei kleinen gleichförmigen Biegungen homogener Metallstäbe. (Diss.) Göttingen, 1907. 8vo. 86 pp.

KOZÁK (J.). Grundprobleme der Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. Vol. II. 1er Teil: Theorie des Schiesswesens auf Grundlage der Wahrscheinlichkeitsrechnung und Fehlertheorie. Wien, Fromme, 1908. 8vo. 21 + 400 pp. M. 16.00

LEBON (E.). Géométrie descriptive et cotée. (Classe de mathématiques A et B.) Paris, Delalain, 1908. 180 pp. Fr. 4.00

LECORNU (L.). Dynamique appliquée (Encyclopédie scientifique, Bibliothèque de mécanique appliquée et Génie, dirigée par M. d'Ocagne). Paris, Doin, 1908. 550 pp. Fr. 5.00

MATRICULATION mechanics papers. London, Clive, 1908. 8vo. 74 pp. 1s.6d.

WOTRUBA (R.). Technische Mechanik der starren, flüssigen und ausdehnsam-flüssigen Körper. Kurzgefasstes Hand- und Hilfsbuch für Techniker und Studierende an mittleren technischen Lehranstalten sowie zum Selbstunterricht. Leipzig, Voigt, 1908. 8vo. 10 + 134 pp. M. 5.50

THE INVERSE OF MEUSNIER'S THEOREM.

BY PROFESSOR EDWARD KASNER.

(Read before the American Mathematical Society, April 25, 1908.)

MEUSNIER'S theorem, relating to the curves drawn on an arbitrary surface

$$\text{(I)} \qquad f(x, y, z) = 0,$$

was extended by Lie * to the curves satisfying any Monge equation of the first order

$$\text{(II)} \qquad f(x, y, z, y', z') = 0,$$

where primes denote differentiation with respect to x. In this note we show that the theorem is valid in the more general case of curves defined by any equation of the form

$$\text{(III)} \qquad Ay'' + Bz'' + C = 0,$$

where A, B, C are arbitrary functions of $x, y, z, y'\, z'$; and that no further extension is possible.

Our problem is to find the most general system of space curves with the *Meusnier property*. This deals with the curvature of the curves of the system which pass through a common point O, in a common direction, and may be stated in any one of the following equivalent ways:

1. The radius of curvature varies as the sine of the angle between the osculating plane and a fixed plane.
2. The circles of curvature generate a sphere.
3. The locus of the centers of curvature is a circle through the point O.
4. The locus of the inverse centers of curvature is a straight line.

It will be convenient to use the last statement, sometimes referred to as Hachette's theorem. Taking the origin at the given point O, and the axis of X along the given direction, we find for the center of curvature

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* *Leipziger Berichte*, vol. 50 (1898), p. 1; *Math. Annalen*, vol. 59 (1904), p. 299.

$$(4) \qquad X_1 = 0, \quad Y_1 = \frac{y''}{y''^2 + z''^2}, \quad Z_1 = \frac{z''}{y''^2 + z''^2}.$$

The inverse center of curvature, *i. e.*, the inverse of this point with respect to a unit sphere about O, is

$$(5) \qquad X = 0, \quad Y = y'', \quad Z = z''.$$

The Meusnier property requires that the point X, Y, Z, for the curves considered, shall be confined to a straight line. Let this line, which is necessarily in the plane normal to the given lineal element, be

$$(6) \qquad X = 0, \quad AY + BZ + C = 0.$$

Then, from (5), the relation between y'' and z'' must be of the form

$$(\text{III}) \qquad Ay'' + Bz'' + C = 0.$$

The line (6) is fixed for a given lineal element, but may vary when the element varies. Hence A, B, C are arbitrary functions of x, y, z, y', z'.

THEOREM. *The system of curves defined by any differential equation of the form (III) possesses the Meusnier property, i. e., the locus of the inverse centers of curvature of those curves of the system which have a lineal element in common is a straight line. Conversely, every system possessing this property satisfies an equation of the form (III).*

Equations of the types I and II give rise by differentiation to special cases of the type III. More generally we may start with any two-parameter family of surfaces

$$(7) \qquad f(x, y, z, a, b) = 0,$$

or any one-parameter family of Monge equations of the first order

$$(8) \qquad f(x, y, z, y', z', a) = 0,$$

and derive equations of the form III. The systems of curves defined by (7) and (8) constitute special species (the second including the first) of the general class III.

The Meusnier property, although stated in metric form, really belongs to the general geometry whose group is composed of all (analytic) point transformations. This follows from the fact that in any point transformation the new values of y'' and z'' are linear and integral in the original values. The type III is, therefore, invariant. The species (7) and (8) are also invariant under this group.

The case where the absolute term C vanishes, so that the equation III reduces to

$$Ay'' + Bz'' = 0, \tag{9}$$

is metrically exceptional. The statement 1. of the Meusnier property then becomes illusory since all curves having an element in common here have a common osculating plane. The sphere of statement 2. reduces to this plane, and the loci of 3. and 4. reduce to the same straight line passing through O. The type (9) gives the most general system of space curves such that curves having an element in common also have a common osculating plane. This type is not invariant under the group of all point transformations but is invariant under the projective group.

In connection with the general type III we may develope a theory of *straightest curves* which includes as special cases the theory of geodesics on a surface and the Hertzian theory of path curves in the non-holonomic motion defined by a Pfaffian equation. A curve of the system represented by an equation III is here termed a straightest curve of the system when its curvature at any point is less than (or at most equal to) the curvature of the other curves of the system which pass through that point in the same direction.

The differential equations of these curves can easily be derived by writing down the conditions that the curvature, which is a known function of x, y, z, y', z', here regarded as fixed, and of y'', z'' regarded as variable, but subject to the relation III, shall be a minimum. We prefer, however, to derive the equations from the fact that the locus of the inverse centers of curvatures is a straight line. When the lineal element has its direction given by any values of y' and z', the associated straight line is found to be

$$\begin{gathered} X + y'Y + z'Z = 0, \\ (Ay' + Bz')X - AY - BZ - \frac{C}{1 + y'^2 + z'^2} = 0. \end{gathered} \tag{10}$$

For a curve of least curvature, the osculating plane, whose general equation is

$$(11)\qquad \begin{vmatrix} X & Y & Z \\ 1 & y' & z' \\ 0 & y'' & z'' \end{vmatrix} = 0,$$

must meet the line (10) in the point nearest to O. This means that (11) must be perpendicular to (10), so that

$$(12)\qquad \begin{vmatrix} Ay' + Bz' & -A & -B \\ 1 & y' & z' \\ 0 & y'' & z'' \end{vmatrix} = 0.$$

Combining this with III, we find, as our final equations,

$$(13)\qquad \begin{aligned} y'' &= \frac{-C(A + Ay'^2 + By'z')}{A(A + Ay'^2 + By'z') + B(B + Bz'^2 + Ay'z')}, \\ z'' &= \frac{-C(B + Bz'^2 + Ay'z')}{A(A + Ay'^2 + By'z') + B(B + Bz'^2 + Ay'z')}. \end{aligned}$$

Our result may be stated as follows:

In any system of curves of type (III) there exists a four-fold infinitude of straightest curves, i. e., curves of least curvature, one passing through each point in each direction. The differential equations of these curves are given by (13).[*]

The systems (13) which thus present themselves are of perfectly general character. For, given any system of ∞^4 space curves, one through each lineal element, its differential equations may be written

$$(14)\qquad y'' = g(x, y, z, y', z'), \quad z'' = h(x, y, z, y', z').$$

By comparing these with (13), the ratios $A : B : C$, and hence an equation of type III, may be found. The inverse problem which thus arises (determination of a system III from its straightest lines) is, in general, capable of only one solution.[†]

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[*] In the case of type I there are ∞^2 such curves, namely the geodesics. In type II there are ∞^3 straightest curves.

[†] An exception arises when the assigned system (14) consists of the straight lines $y'' = 0$, $z'' = 0$. The system III may then be any one of the form (9).

It is of interest to notice that the type III presented itself first in connection with the general transformation theory * of the lineal elements of space. Such a transformation is defined by setting $x_1, y_1, z_1\ y_1', z_1'$ equal to arbitrary functions of x, y, z, y', z'. Lie showed that the only case in which every union is turned into a union is the extended point transformation. There may, however, be other transformations which convert some unions into unions. Examples of this sort have been given by Lie in which the unions considered are the curves of a minimal complex or some other Monge equation (II). The general result is as follows:

The most extensive systems of curves that can be converted into curves by element transformations which are not merely extended point transformations are those defined by differential equations of type III, i. e., precisely those characterized by the Meusnier property.

For any system of type III transformations of this sort may be found. If a system not satisfying an equation III is transformed into curves, the element transformation is necessarily an extended point transformation.

COLUMBIA UNIVERSITY.

ON THE DISTANCE FROM A POINT TO A SURFACE.†

BY PROFESSOR PAUL SAUREL.

(Read before the American Mathematical Society, April 25, 1908.)

IN a recent number of the BULLETIN,‡ Professor Hedrick has called attention to the fact that the normal distance from a given point to a surface may be a minimum among the distances in every normal plane section without however being a minimum among the distances to the surface. In this connection it may be of interest to observe that for surfaces of a very general type the phenomenon in question can occur only when the given point is a principal center of curvature. In fact, if the equation of the surface in the neighborhood of the foot of the normal can be written in the form

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* Cf. abstract of the author's paper on this subject in the BULLETIN, vol. 10 (1904), p. 492.

† The theorems established in a previous note (BULLETIN, vol. 13, p. 447) are not new, as I then believed; they are to be found in section 60 of Goursat's Cours d'Analyse. I take this opportunity of acknowledging Professor Hedrick's kindness in calling my attention to the fact.

‡ BULLETIN, April, 1908, p. 321.

$$(1) \qquad z = ax^2 + by^2 + R,$$

in which the remainder R is of the third or higher order in x and y, and a and b are not both equal to zero, it is easy to establish the following theorems:

I. If the foot of the normal be an elliptic point of the surface, the distance from the nearer (more remote) of the principal centers of curvature may be a minimum (maximum) among the distances in every normal section without being a minimum (maximum) among the distances to the surface.

As illustrations of these two statements may be mentioned the distances from the point $(0, 0, \frac{1}{2})$ to the surfaces

$$z = x^2 + \frac{y^2}{2} + \frac{3}{2}x^2y, \quad \text{and} \quad z = x^2 + 2y^2 - 3x^2y + 3x^4.$$

II. If the foot of the normal be a hyperbolic point of the surface, the distance from a principal center of curvature may be a minimum among the distances in every normal section without being a minimum among the distances to the surface.

The distance from the point $(0, 0, \frac{1}{2})$ to the surface

$$z = x^2 - y^2 + 6x^2y - 3x^4$$

furnishes an illustration of this statement.

III. If the foot of the normal be a parabolic point of the surface, the distance from the principal center of curvature may be a minimum among the distances in every normal section without being a minimum among the distances to the surface.

In this statement the term parabolic point is to be understood as meaning any point for which a or b in equation (1) vanishes.

Professor Hedrick has given an example that falls under this theorem. An equally simple example is furnished by the distance from the point $(0, 0, \frac{1}{4})$ to the surface

$$z = 2x^2 + 10x^2y.$$

IV. The phenomenon discovered by Professor Hedrick cannot occur unless the given point be a principal center of curvature.

To establish the last of these theorems it is sufficient to observe that the square of the distance from a point $(0, 0, d)$ on the normal to a point (x, y, z) on the surface is given by an expression of the form

$$d^2 + Ax^2 + By^2 + \cdots,$$

in which A and B are constants and the omitted terms are of the third order at least, and that when the given point is not a principal center of curvature the coefficients A and B do not vanish. If, however, the given point be a principal center of curvature, A or B will vanish and an examination of the sign of the non-vanishing coefficient will easily yield the first three theorems.

NEW YORK,
April 7, 1908.

ON THE SOLUTION OF ALGEBRAIC EQUATIONS IN INFINITE SERIES.

BY PROFESSOR P. A. LAMBERT.

(Read before the American Mathematical Society, April 25, 1908.)

I. *Introduction.*

The purpose of this paper is to present a general method for determining all the roots of any algebraic equation by means of infinite series. The method consists in forming three algebraic functions of x from the given equation

$$(1) \qquad f(y) = 0,$$

(a) by introducing a factor x into all the terms of (1) except the first and last;

(b) by introducing a factor x into all the terms of (1) except the first and second;

(c) by introducing a factor x into all the terms of (1) except the second and last.

These algebraic functions are expanded into power series in x by Laplace's series. If in these power series x is made unity, the resulting series, if convergent, determine the roots of the given equation. It will be shown that all the roots of the given equation can be expressed in infinite series derived either from the algebraic function formed in accordance with (a), or from the two algebraic functions formed in accordance with (b) and (c). The method presupposes the solution of the two-term equation

$$ay^n + e = 0. \tag{2}$$

In a paper entitled "New applications of Maclaurin's series in the solution of equations" read before the American Philosophical Society, April 3, 1903, the author applied this method to numerical equations. After completing that paper, the author found in an extract of a letter from Cauchy to Coriolis, of January 29, 1837, published in the *Comptes Rendus* of the Paris Academy, an announcement of important results to be obtained by breaking up an equation into two parts and introducing as a factor into one part a parameter which is ultimately to be made unity. In a postscript Cauchy states that he discovered the advantage of making one part a binomial. The author has not been able to find the method sketched in this letter developed.

II. *The Three-Term Equation.*

From the three-term equation

$$ay^n + by^k + e = 0, \tag{3}$$

in accordance with (a), (b), (c) of Article I, are formed the three algebraic functions of x

$$ay^n + by^k x + e = 0, \tag{4}$$

$$ay^n + by^k + ex = 0, \tag{5}$$

$$ay^n x + by^k + e = 0. \tag{6}$$

If equations (4), (5), (6) are written in the form

$$y = \left[-\frac{e}{a} - x\frac{b}{a}y^k \right]^{\frac{1}{n}}, \tag{7}$$

$$y = \left[-\frac{b}{a} - x\frac{e}{a}y^{-k} \right]^{\frac{1}{n-k}}, \tag{8}$$

$$y = \left[-\frac{e}{b} - x\frac{a}{b}y^n \right]^{\frac{1}{k}}, \tag{9}$$

the algebraic functions y can be expanded into power series in x by means of Laplace's series,* which asserts that if

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* *Mémoires de l'Académie de Sciences de Paris*, 1777.

$$y = f[z + x\phi(y)], \tag{10}$$

$$\begin{aligned} F(y) = F\{f(z)\} &+ x\phi\{f(z)\}\frac{d}{dz}F\{f(z)\} \\ &+ \frac{x^2}{2\,!}\frac{d}{dz}\left[\overline{\phi\{f(z)\}}^{2}\frac{d}{dz}F\{f(z)\}\right] + \cdots \\ &+ \frac{x^n}{n\,!}\frac{d^{n-1}}{dz^{n-1}}\left[\overline{\phi\{f(z)\}}^{n}\frac{d}{dz}F\{f(z)\}\right] + \cdots. \end{aligned} \tag{11}$$

If the algebraic function y of equation (7) is expanded into a power series in x by Laplace's series and x is made unity in this series, and if y_0 denotes any one of the n roots of the two-term equation

$$ay^n + e = 0,$$

there results *

$$\begin{aligned} y = y_0 &+ \frac{b}{ne}y_0^{1+k} + \frac{1+2k-n}{2\,!}\frac{b^2}{n^2e^2}y_0^{1+2k} \\ &+ \frac{(1+3k-n)(1+3k-2n)}{3\,!}\frac{b^3}{n^3e^3}y_0^{1+3k} + \cdots \\ &+ \frac{(1+sk-n)(1+sk-2n)\cdots[1+sk-(s-1)n]}{s\,!}\frac{b^s}{n^se^s}y_0^{1+sk} + \cdots. \end{aligned} \tag{12}$$

If the first n terms of series (12) are placed in a row in regular order, and the following n terms are placed under the terms of this row in regular order, and so on indefinitely, the terms of series (12) will be arranged in n partial infinite series composed respectively of the terms standing in each of the n columns.

This rearrangement of terms is permissible inasmuch as it will appear subsequently that this series is used only when it is absolutely convergent.

The terms of the left-hand partial series are

$$\begin{aligned} &y_0 \\ &+ y_0^{1+nk}\frac{b^n}{n^ne^n}\frac{(1+nk-n)(1+nk-2n)\cdots[1+nk-(n-1)n]}{n\,!} \end{aligned}$$

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* Laplace's series is used only as a matter of convenience. The same expansion can be found by using Lagrange's series, Maclaurin's series, or the multinomial theorem by assuming $y = a_0 + a_1x + a_2x^2 + \cdots + a_tx^t + \cdots$ in equation (7).

$$+y_0^{1+2nk}\frac{b^{2n}}{n^{2n}e^{2n}}\,\frac{(1+2nk-n)(1+2nk-n)\cdots[1+2nk-(2n-1)n]}{(2n)\,!}$$

$$(13)\quad + \quad \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

$$+\,y_0^{1+tnk}\frac{b^{tn}}{n^{tn}e^{tn}}\,\frac{(1+tnk-n)(1+tnk-2n)\cdots[1+tnk-(tn-1)n]}{tn\,!}$$

$$+y_0^{1+(t+1)nk}\frac{b^{(t+1)n}}{n^{(t+1)n}e^{(t+1)n}}\,\frac{[1+(t+1)nk-n]\,[1+(t+1)nk-2n]\cdots[1+(t+1)nk-\{(t+1)n-1\}n]}{[(t+1)n]\,!}$$

$$+\quad \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

Cauchy's ratio test for convergency is applied to series (13). The ratio of the general term of (13) to the preceding term is

$$(14)\quad y_0^{nk}\frac{b^n}{n^ne^n}\,\frac{[1+tnk+(k-1)n]\cdots[1+tnk]\,[1-t(n-k)n]\cdots[1-t(n-k)n-(n-k-1)n]}{(tn+1)(tn+2)\cdots(t+1)n}.$$

In each of the two groups of factors in the numerator of the last fraction in this ratio the successive factors differ by $-n$. The first group contains k factors, the second group contains $n-k$ factors.

If the limiting value of this ratio (14) when t becomes infinitely large is numerically less than unity, series (13) is absolutely convergent. This condition gives the inequality between absolute values

$$(15)\qquad \frac{e^k}{a^k}\,\frac{b^n}{n^ne^n}\,k^k(n-k)^{n-k}<1,$$

whence

$$(16)\qquad \left|\frac{b^n}{a^ke^{n-k}}\right|<\frac{n^n}{k^k(n-k)^{n-k}}.$$

In like manner it is proved that the inequality (16) is the condition for the absolute convergence of each one of the remaining $n-1$ partial series into which the infinite series (12) has been broken up.

It follows that when condition (16) is satisfied the infinite series (12) is absolutely convergent.

From equation (7), by means of Laplace's series (11), power series in x can be found for y^n and y^k. If in these power series x is made unity, the resulting series for y^n and y^k are found to be absolutely convergent when condition (16) is satisfied. The values of y^n and y^k make equation (3) an identity. Therefore the value of y in (12) is not only convergent when (16) is satisfied, but it actually represents a root of equation (3). In fact this expression for y determines the n roots of equation (3).

In like manner the series which express the roots of the three-term equation (3) derived from the power series obtained from equations (8) and (9) by Laplace's series are found to be absolutely convergent provided

$$(17)\qquad \left|\frac{b^n}{a^k e^{n-k}}\right| > \frac{n^n}{k^k(n-k)^{n-k}}.$$

When condition (17) is satisfied the series obtained from (8) determines $n-k$, the series obtained from (9) determines k roots of equation (3).

Raabe's convergency test* shows that the series which express the roots of equation (3) derived from equations (7), (8), (9) are all convergent for the limiting condition

$$(18)\qquad \left|\frac{b^n}{a^k e^{n-k}}\right| = \frac{n^n}{k^k(n-k)^{n-k}}.$$

It follows that all the roots of the three-term equation (3) can always be expressed in infinite series derived either from equation (4) or from the two equations (5) and (6).

Infinite series for determining all the roots of the three-term equation, differing from series (12) only in form, are obtained by Nekrasoff, *Mathematische Annalen*, volume 29, by means of evaluating a definite integral, and by McClintock, in volume 17 of the *American Journal of Mathematics*, by means of his calculus of enlargement. Nekrasoff and McClintock both recognize that their series can be obtained by applying Lagrange's series. Both investigators found convergency conditions corresponding to (16) and (17) by the methods used in this article. Condition (18) seems to be new.

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* The infinite series $\sum_{n=1}^{n=\infty} u_n$ is convergent if

$$\lim_{n=\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}>1,$$

Zeitschrift für Mathematik und Physik, vol. 10 (1832).

The results obtained by applying the method of Article I for finding infinite series determining the roots of equations of more than three terms in the following articles, so far as the author knows, are new..

III. *The Four-Term Equation.*

From the four-term equation

$$(19) \qquad ay^n + by^k + cy^l + e = 0$$

in accordance with (*a*), (*b*), (*c*) of Article I are formed the three equations

$$(20) \qquad ay^n + by^k x + cy^l x + e = 0,$$

$$(21) \qquad ay^n + by^k + cy^l x + ex = 0,$$

$$(22) \qquad ay^n x + by^k + cy^l x + e = 0,$$

defining y as algebraic functions of x.

The equations (20), (21), (22) may be written in the form

$$(23) \qquad y = \left[-\frac{e}{a} - x\left(\frac{b}{a}y^k + \frac{c}{a}y^l \right) \right]^{1/n},$$

$$(24) \qquad y = \left[-\frac{b}{a} - x\left(\frac{c}{a}y^{l-k} + \frac{e}{a}y^{-k} \right) \right]^{1/(n-k)},$$

$$(25) \qquad y = \left[-\frac{e}{b} - x\left(\frac{a}{b}y^k + \frac{c}{b}y^l \right) \right]^{1/k}.$$

If the y of equation (23) is expanded into a power series in x by Laplace's series, and after x is made unity the terms of the resulting series are arranged in groups according to the ascending powers of

$$(26) \qquad \frac{c}{ne}\left(-\frac{e}{a} \right)^{l/n},$$

it will be found that the terms of the first group constitute the series which expresses the roots of the three-term equation (3) provided condition (16) is satisfied, and that all the successive groups are convergent when condition (16) is satisfied.

It follows that the series which expresses the roots of equation (19) derived from equation (23) can be written

$$(27)\quad y = X_0 + X_1 \frac{c}{ne}\left(-\frac{e}{a}\right)^{l/n} + X_2 \frac{c^2}{n^2e^2}\left(-\frac{e}{a}\right)^{2l/n} + X_3 \frac{c^3}{n^3e^3}\left(-\frac{e}{a}\right)^{3l/n} + \cdots,$$

where X_0, X_1, X_2, X_3, X_4, $\cdots$ represent convergent infinite series.

The series (27) is convergent provided

$$(28)\qquad \left|\frac{c}{ne}\left(-\frac{e}{a}\right)^{l/n}\right| < 1.$$

Condition (28), when both members of the inequality are affected by the exponent n, becomes

$$(29)\qquad \left|\frac{c^n}{a^le^{n-l}}\right| < n^n.$$

The conditions

$$(30)\qquad \left|\frac{b^n}{a^ke^{n-k}}\right| \leqq \frac{n^n}{k^k(n-k)^{n-k}}, \qquad \left|\frac{c^n}{a^le^{n-l}}\right| < n^n$$

are therefore sufficient for the absolute convergence of the infinite series expressing the roots of the four-term equation (19) derived from equation (23).

In like manner it is found that the conditions sufficient for the absolute convergence of the infinite series expressing the roots of the four-term equation (19) derived from equations (24) and (25) are

$$(31)\qquad \left|\frac{b^n}{a^ke^{n-k}}\right| \geqq \frac{n^n}{k^k(n-k)^{n-k}}, \qquad \left|\frac{b^{n-l}}{a^{k-l}c^{n-k}}\right| > \frac{1}{(n-k)^{n-k}},$$

$$(32)\qquad \left|\frac{b^n}{a^ke^{n-k}}\right| \geqq \frac{n^n}{k^k(n-k)^{n-k}}, \qquad \left|\frac{c^k}{b^le^{k-l}}\right| < k^k.$$

Either the first condition of (30) or the first condition of (31) and (32) must be satisfied. If it is possible to show that when the first condition of (30) or of (31) and (32) is satisfied the second inequalities can always be satisfied, the possibility of determining the roots of the four-term equation by means of infinite series is established.

The substitution

$$(33)\qquad y = z^l,$$

where s is a positive integer, transforms the four-term equation (19) into the four-term equation

$$(34) \qquad ay^{ns} + by^{ks} + cy^{ls} + e = 0.$$

The convergency conditions for equation (34) corresponding to conditions (30), (31), (32) are

$$(35) \quad \left|\frac{b^n}{a^k e^{n-k}}\right| \leqq \frac{n^n}{k^k(n-k)^{n-k}}, \quad \left|\frac{c^n}{a^l e^{n-l}}\right| < s^n n^n,$$

$$(36) \quad \left|\frac{b^n}{a^k e^{n-k}}\right| \geqq \frac{n^n}{k^k(n-k)^{n-k}}, \quad \left|\frac{b^{n-l}}{a^{k-l} c^{n-k}}\right| > \frac{1}{s^{n-k}(n-k)^{n-k}},$$

$$(37) \quad \left|\frac{b^n}{a^k e^{n-k}}\right| \geqq \frac{n^n}{k^k(n-k)^{n-k}}, \quad \left|\frac{c^k}{b^l e^{k-l}}\right| < s^k k^k.$$

The value of s can always be taken so large that the second inequality of (35) is satisfied, or that the second inequalities of (36) and (37) are satisfied simultaneously.

When conditions (35) are satisfied, all the roots of equation (34) are expressed by the infinite series derived from the equation

$$(38) \qquad ay^{ns} + by^{ks}x + cy^{ls}x + e = 0;$$

when conditions (36) and (37) are satisfied, all the roots of equation (34) are expressed by the infinite series derived from the equations

$$(39) \qquad ay^{ns} + by^{ks} + cy^{ls}x + ex = 0,$$

$$(40) \qquad ay^{ns}x + by^{ks} + cy^{ls}x + e = 0.$$

It is therefore always possible to express all the roots of equation (34) in infinite series. The roots of equation (19) are found from the roots of equation (34) by substituting in (33).

IV. *The Five-Term Equation.*

From the five-term equation

$$(41) \qquad ay^n + by^k + cy^l + dy^m + e = 0,$$

in accordance with (*a*), (*b*), (*c*) of Article I, are formed the three equations

$$ay^n + by^kx + cy^lx + dy^mx + e = 0, \tag{42}$$

$$ay^n + by^k + cy^lx + dy^mx + ex = 0, \tag{43}$$

$$ay^nx + by^k + cy^lx + dy^mx + e = 0, \tag{44}$$

defining y as algebraic functions of x.

The equations (42), (43), (44) may be written in the form

$$y = \left[-\frac{e}{a} - x\left(\frac{b}{a}y^k + \frac{c}{a}y^l + \frac{d}{a}y^m\right)\right]^{\frac{1}{n}}, \tag{45}$$

$$y = \left[-\frac{b}{a} - x\left(\frac{a}{a}y^{l-k} + \frac{d}{a}y^{m-k} + \frac{e}{a}y^{-k}\right)\right]^{\frac{1}{n-k}}, \tag{46}$$

$$y = \left[-\frac{e}{b} - x\left(\frac{a}{b}y^n + \frac{c}{b}y^l + \frac{d}{b}y^m\right)\right]^{\frac{1}{k}}. \tag{47}$$

If the y of equation (45) is expanded into a power series in x by Laplace's series and after x is made unity in this power series the terms of the resulting series are arranged in groups according to the ascending powers of

$$\frac{d}{ne}\left(-\frac{e}{a}\right)^{\frac{m}{n}}, \tag{48}$$

it will be found that the terms of the first group constitute the series which expresses the roots of the four-term equation (19) provided the conditions (30) are satisfied, and that all the successive groups are convergent when the conditions (30) are satisfied.

It follows that the series which expresses the roots of equation (41) derived from equation (42) can be written

$$\begin{aligned} y = Y_0 + Y_1\frac{d}{ne}\left(-\frac{e}{a}\right)^{\frac{m}{n}} + Y_2\frac{d^2}{n^2e^2}\left(-\frac{e}{a}\right)^{\frac{2m}{n}} \\ + Y_3\frac{d^3}{n^3e^3}\left(-\frac{e}{a}\right)^{\frac{3m}{n}} + \cdots, \end{aligned} \tag{49}$$

where $Y_0, Y_1, Y_2, Y_3, \cdots$, represent convergent infinite series.

The series (49) is convergent provided

$$\left|\frac{d}{ne}\left(-\frac{e}{a}\right)^{\frac{m}{n}}\right| < 1. \tag{50}$$

Condition (50), when both members of the inequality are affected by the exponent n, becomes

$$(51)\qquad \left|\frac{d^n}{a^m e^{n-m}}\right| < n^n.$$

The conditions

$$(52)\qquad \left|\frac{b^n}{a^k e^{n-k}}\right| \leqq \frac{n^n}{k^k(n-k)^{n-k}}, \quad \left|\frac{c^n}{a^l e^{n-l}}\right| < n^n, \quad \left|\frac{d^n}{a^m e^{n-m}}\right| < n^n$$

are therefore sufficient for the absolute convergence of the series expressing the roots of the five-term equation (41) derived from equation (42).

In like manner it is found that the conditions sufficient for the absolute convergence of the infinite series expressing the roots of the five-term equation (41) derived from equations (43) and (44) are

$$(53)\qquad \begin{gathered} \left|\frac{b^n}{a^k e^{n-k}}\right| \geqq \frac{n^n}{k^k(n-k)^{n-k}}, \quad \left|\frac{b^{n-l}}{a^{k-l} c^{n-k}}\right| > \frac{1}{(n-k)^{n-k}}, \\ \left|\frac{b^{n-m}}{a^{k-m} d^{n-k}}\right| > \frac{1}{(n-k)^{n-k}}, \end{gathered}$$

$$(54)\qquad \left|\frac{b^n}{a^k e^{n-k}}\right| \geqq \frac{n^n}{k^k(n-k)^{n-k}}, \quad \left|\frac{c^k}{b^l e^{k-l}}\right| < k^k, \quad \frac{d^k}{b^m e^{k-m}} < k^k.$$

Either the first condition of (52), or the first conditions of (53) and (54) must be satisfied.

The substitution

$$(55)\qquad y = z^s$$

transforms the five-term equation (41) into the five-term equation

$$(56)\qquad ay^{ns} + by^{ks} + cy^{ls} + dy^{ms} + e = 0.$$

It is always possible to determine s so that either the convergency conditions for equation (56) corresponding to (52), or those corresponding to (53) and (54) are satisfied. It is therefore always possible to determine the roots of equation (56) by infinite series. The roots of the five-term equation (41) are found from the roots of (56) by substituting in (55).

V. *Conclusion.*

The method used to set up the convergency conditions for the infinite series expressing the roots of the four-term equation derived from the equations formed in accordance with (*a*), (*b*) (*c*) of Article I when the convergency conditions of the three-term equation are known, and to set up the convergency conditions for the five-term equation when the convergency conditions for the four-term equation are known, can be used to set up the convergency conditions for the t-term equation when the convergency conditions for the $(t-1)$-term equation are known.

In fact, the convergency conditions for an equation of any number of terms can be written mechanically.

For the t-term equation

$$f(y) = 0 \tag{57}$$

it is always possible to determine the s of the substitution

$$y = z^s \tag{58}$$

so that the convergency conditions of the infinite series expressing the roots of the t-term equation

$$f(z^s) = 0 \tag{59}$$

derived from the equation formed from (59) in accordance with (*a*) of Article I, or derived from the two equations formed from (59) by (*b*) and (*c*) of Article I are satisfied. The roots of the t-term equation (57) are found from the roots of the equation (59) by substituting in (58).

It is therefore always possible to find all the roots of an algebraic equation by means of infinite series.

LEHIGH UNIVERSITY,
April, 1908.

THE DEDUCTION OF THE ELECTROSTATIC EQUATIONS BY THE CALCULUS OF VARIATIONS.

BY DR. ARTHUR C. LUNN.

(Read before the Chicago Section of the American Mathematical Society, April 17, 1908.)

THE construction of a mathematical theory of classes of physical phenomena for which no detailed mechanical explanation is available has in several instances led to the enunciation of some general minimal property, usually relating to the distribution of energy and analogous to the principles of least action and of virtual velocities in dynamics. Such a variational "principle" may be regarded either as a concise equivalent for a system of algebraic or differential equations already obtained, or as a form of hypothesis from which those equations may be deduced. The systematic use in the latter sense by Lagrange of the principle of virtual velocities has given his Mécanique Analytique a unity and simplicity far beyond those of most later writings on dynamics.

Corresponding to the equations of electrostatic equilibrium, a number of related forms have been given of a minimizing condition on the energy. These differ among themselves mainly in the character of the variations implied, whether for example variations of a field in which there are charges given or of the charges in a given field.* But in all the forms known to the writer the existence of the electrostatic potential is assumed, while from the point of view of the theory of vector fields it seems more fitting to avoid this assumption if possible, since it is only a special kind of vector field which has a scalar potential. The following deduction according to the formal methods of the calculus of variations, showing how the potential function may appear simply as a lagrangian multiplier, is analogous to the method used by Lagrange for the theory of incompressible fluids, in which the hydrostatic pressure is not introduced at the start as a physical concept, but appears as the multiplier corresponding to the differential condition of invariable volume.†

Adopting the point of view of Faraday and Maxwell, let it be supposed that the electrostatic phenomena can be completely described in terms of a vector point function **E**, the electric force on unit charge, and a scalar point function ϵ, the dielectric coefficient;‡ that in terms of these the displacement and the volume and surface densities of charge are

$$(1) \qquad \mathbf{D} = \epsilon\mathbf{E}, \quad \rho = \operatorname{div} \mathbf{D}, \quad \sigma = \mathbf{n}'\mathbf{D}' + \mathbf{n}''\mathbf{D}'',$$

where $\mathbf{n}', \mathbf{n}''$ are opposite unit normals of a surface on which the surface density is the discontinuity in the normal component of $\mathbf{D}$; and that the total energy stored in a given volume by means of the electrostatic strain is

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* See for example: Webster, The Theory of Electricity and Magnetism, § 131; Weber, Differentialgleichungen der mathematischen Physik, I, p. 310.

† Mécanique Analytique, part I, sec. VII.

‡ The vector notation and units are those of Heaviside, Electromagnetic Theory.

$$W = \tfrac{1}{2}\int \mathbf{E}\mathbf{D}dv. \tag{2}$$

If the total charge on a conductor be supposed to include the surface charge on the boundary between conductor and dielectric, then the charge belonging to any conductor with index i is

$$Q_i = \int \operatorname{div} \mathbf{D}dv_i + \int (\mathbf{n}'\mathbf{D}' + \mathbf{n}''\mathbf{D}'')dS_i,$$

but by transformation of the volume integral to a surface integral this becomes

$$Q_i = \int \mathbf{n}'\mathbf{D}'dS_i, \tag{3}$$

where $\mathbf{n}'$ is the unit normal into the dielectric. If $\mathbf{D}$ should have a discontinuity at a surface within the conductor, the terms in the surface integrals belonging to opposite sides of that surface would be cancelled by corresponding terms from the volume integral, so that the integration need cover only the bounding surface. It is simpler, however, to assume that ϵ and therefore $\mathbf{D}$ are zero everywhere within the conductor, so that the only charge is on the bounding surface, in which case formula (3) still holds.

If ϵ be supposed to vanish within the substance of a conductor and to be a definite physical coefficient different from zero at each point of the dielectric, then the form here to be considered of the condition of equilibrium is that the first variation of W shall vanish for all variations of $\mathbf{E}$ consistent with giving assigned values to the volume and surface densities in the dielectric and to the total charge on each conductor; or, analytically, that with the integration extended throughout the dielectric

$$\delta\int \epsilon\mathbf{E}^2dv = 0 \tag{4}$$

for all variations $\delta\mathbf{E}$ satisfying the conditions

$$div(\epsilon\mathbf{E}) = \rho, \tag{I}$$

where ρ is an assigned function throughout the volume of the dielectric; and

$$\text{(II)} \qquad \epsilon'\mathbf{E}'\mathbf{n}' + \epsilon''\mathbf{E}''\mathbf{n}'' = \sigma,$$

where σ is an assigned function over all su
uity in the dielectric; and

$$\text{(III)} \qquad \int \epsilon'\mathbf{E}'\mathbf{n}'dS_i = Q_i,$$

where each Q_i is an assigned constant for
conductor.

To this conditioned variation of W there
ing to Lagrange's method of multipliers a f
modified integral

$$\text{(5)} \qquad U = W - \int V \operatorname{div}(\epsilon\mathbf{E})dv - \sum V$$

where the undetermined function V is th
sponding to the differential condition (I) a
are the multipliers corresponding to the
(III). But

$$\int V \operatorname{div}(\epsilon\mathbf{E})dv = \int \operatorname{div}(V\epsilon\mathbf{E})dv -$$

and

$$\int \operatorname{div}(V\epsilon\mathbf{E})dv = -\sum\int V'\epsilon'\mathbf{E}'dS_i - \int(V'\epsilon'$$

where S_0 denotes the surfaces of discontinu
which in the integration by parts must be
the surfaces of the conductors as the bound
integration. Hence U can be written

$$\text{(6)} \quad U = \int \epsilon(\tfrac{1}{2}\mathbf{E}^2 + \mathbf{E}\nabla V)dv + \sum\int(V' -$$
$$+ \int(V'\epsilon'$$

and the condition $\delta U = 0$ for all admiss
gives the required equations.

By variations $\delta\mathbf{E}$ which vanish at the su
first term only of U is affected and
$\mathbf{E} = -\nabla V$, so that the vector function $\mathbf{E}$
potential V, which according to (I) must sa
Poisson equation

$$\text{(7)} \qquad \operatorname{div}(\epsilon\nabla V) + \rho = 0$$

Then with this condition satisfied variations of E which vanish at S_0 but not at the surfaces S_i affect only the second term of U and give the condition $V = V_i$, making the potential constant over the surface of each single conductor; and finally variations which at the surfaces S_0 are restricted only by the boundary condition (II) give the condition $V' = V''$, making the potential continuous even at surfaces where its derivatives may be discontinuous.

CHICAGO, ILL.,
May, 1908.

THE FOURTH INTERNATIONAL CONGRESS OF MATHEMATICIANS.

THE fourth international congress of mathematicians was held at Rome, April 6 to 11, 1908, under the efficient management of the Circolo Matematico di Palermo and under the patronage of His Majesty, the King of Italy. Including the ladies who accompanied the members of the congress, the enrollment was more than seven hundred. The list contains the names of the following Americans: Miss E. M. Coddington, H. W. Curjel, E. W. Davis, T. S. Fiske, A. B. Frizell, W. J. Graham, J. G. Hardy, E. A. Harrington, A. S. Hawkesworth, T. F. Holgate, A. Macfarlane, Artemas Martin, C. L. E. Moore, E. H. Moore, Simon Newcomb, G. D. Olds, G. B. Pegram, D. E. Smith, J. M. Van Vleck, W. D. A. Westfall.

The general order of the program provided for sectional meetings in the morning and general conferences in the afternoon. However this order was broken occasionally. The arrangements of the committee on entertainment left nothing to be desired.

The first meeting of the members of the congress was at the reception offered by Professor Tonelli, Rector of the University of Rome, Sunday evening, April 5. This was the beginning of the social part of the occasion. Those who attended the congress will always have pleasant recollections of these receptions and other entertainments. The opening reception was held in the library of the University. The Mayor of the city of Rome, Mr. Nathan, was present. Acquaintances which were to broaden during the week began here. Excellent refreshments were served, and the mathematicians showed that they were not incapable of enjoying this part of the program.

The congress officially opened on Monday, April 6, at a meeting held at the Capitol, in the presence of His Majesty, King Vittorio Emanuele III. The meeting was opened by Mayor Nathan, who welcomed the congress in the name of the city of Rome. The minister of public instruction, Rava, extended a welcome on behalf of the government of Italy. President Blaserna of the committee also made an appropriate address. The session closed with the address of Professor V. Volterra on: "Le matematiche in Italia nella seconda metà del secolo XIX."

The lecturer began by recalling the period of Italian political rejuvenation, in which the whole life of the nation was renewed and the universities reopened. It was then that the instruction in higher mathematics was instituted and entrusted to such mathematicians as Brioschi, Betti, Cremona, Fergola, and Battaglini. Very soon after these followed other mathematicians of no less eminence. The period of original and scientific research in Italy may be said to date from that time. From that moment different Italian schools of mathematics began to be created and to develope.

After drawing a brief comparison between the Italian studies in the first and the second half of the nineteenth century, in order to throw light on the substantial diversity of the surroundings in which they developed, the lecturer passed in review the different Italian mathematical schools, showing their tendencies and their scope.

On account of their research in mathematical physics, which principally absorbed the attention of Betti and Beltrami, he called them the champions of mathematical physics in Italy. He examined the advances in electricity initiated by Betti and continued by a series of other mathematicians, and the researches of Beltrami, who can be placed among those who investigated in a systematic manner the perturbations introduced into the equations of mathematical physics by the hypothesis of a curvature of space.

After having mentioned other investigators in mathematical physics and mechanics, the lecturer passed to another order of investigation which was developed in Italy, viz., that of the theory of functions and allied subjects in analysis. He recalled, in this connection, his address at the Paris Congress on Betti, Brioschi, Casorati, and on the different methods in which these three mathematicians (who were the initiators in Italy of the study of the theory of functions) conceived the theory itself. He recalled also the various investigators of analysis and dwelt more especially on a branch of research which flourished in Italy a little apart, and which was somewhat forgotten for some years, but which recently acquired general interest and attention, that is research in functions of real variables and their singularities. Dini introduced and diffused their study in Italy and put this theory at the basis of his instruction in the infinitesimal calculus. It then took a double direction, one conducted to research in the field of pure mathematics while the other led to studies which acquired a more philosophic character.

From these researches the lecturer passed to those which, in Italy, are wont to be called geometric. He spoke of the singular struggle, which manifested itself more acutely in Italy than elsewhere, between those who were called analysts and those who were called geometers, but demonstrated the unreality of the distinction which was usually made, and the equivocation on which it was founded. In these days the distinction can only be called a relic of the past.

He examined next the work of Cremona and of his students and disciples, and observed that, forty years after Cremona had begun his teaching, Klein affirmed that Italy had become the real center of geometric research, although this field had been almost ignored in Italy until Cremona initiated his course in higher geometry. The lecturer also examined the researches in hypergeometry, algebraic geometry, and especially the geometric interpretation of the theory of forms. He showed how the most recent Italian geometric investigations, being connected with those of Picard on the theory of algebraic functions, reenter the sphere of the theory of functions. He spoke also of infinitesimal geometry, in which much research has been carried on in Italy, and which makes a noble counterpart to the work in pure and algebraic geometry.

Finally, the history of mathematics developed in the last fifty years was recalled, and the critical publications on the works of Galileo were mentioned.

The lecturer closed by showing that the present time in Italy presents some analogies with a half century ago, and that now, as then, a reform in the program of mathematical studies is necessary. These, in fact, in Italy, are identified with the schools of engineering, for which he urged a more modern organization. He concluded by expressing the hope for a continuous development in Italian mathematical thought harmoniously united with that of other nations.

In the afternoon the congress met at the Palazzo Corsini and organized. Professor Blaserna was chosen as permanent president, after which the various other officers were named. The first business of the session was the report, presented by Professor Segre, of the committee on the Guccia Medal. There were three competing memoirs, but for various reasons these were adjudged unworthy of the prize. The committee therefore examined the literature published during the time from November 1, 1904, to July 1, 1907, and decided that the prize should go to Professor Francesco Severi for his investigations on the geometry on an algebraic surface. The publications of Professor Severi referred to are the following:

1. "Sulle superficie algebriche che posseggano integrali di Picard della 2^a specie" (*Mathematische Annalen*, volume 61).

2. "Sulla differenza tra i numeri degli integrali di Picard della 1^a e 2^a specie appartenenti ad una superficie algebrica" (*Atti di Torino*, volume 40).

3. "Sulla totalità delle curve algebriche tracciate sopra una superficie algebrica" (*Mathematische Annalen*, volume 62).

4. "Il teorema d'Abel sulle superficie algebriche" (*Annali di Matematica*, series 3, volume 12).

5. "Intorno al teorema d'Abel sulle superficie algebriche ed alle riduzioni a forma normale degl' integrali di Picard" (*Rendiconti di Palermo*, volume 21).

6. "Sul teorema di Riemann-Roch e sulle serie continue di curve appartenenti ad una superficie algebrica" (*Atti di Torino*, volume 40).

7. "Sulle curve algebriche virtuali appartenenti ad una superficie algebrica" (*Rendiconti del R. Istituto Lombardo*, series 2, volume 40).

8. "Osservazioni varie di geometria sopra una superficie algebrica o sopra una varietà" (*Atti del R. Istituto Veneto*, volume 65).

After the reading of the report the Guccia Medal was presented to Professor Severi.

Professor Mittag-Leffler then read his address "Sur la représentation arithmétique des fonctions analytiques générales d'une variable complexe."

He began by recalling that the central point of the theory of analytic functions is, with Weierstrass, the power series

$$p(x) = c_0 + c_1x + c_2x^2 + \cdots$$

and that this series is the source from which flows, successively and entirely by transformation and continuation, the analytic function in its entirety. He recalled the theorem of Weierstrass: If an analytic relation, however general or however special, exists between several different power series or their derivatives, this same relation subsists also for the functions in their totality.

The address had for its object to express the principal solutions, obtained during the last ten years, of the following problem: "To form arithmetic expressions of a variable x and of an infinite sequence of constants $c_0, c_1, c_2, \cdots$ which are linear in these constants and have the property of representing the given function $F(x)$, in a domain in which the $c_0, c_1, c_2 \cdots$, once determined, are defined in a unique manner." The first efforts toward the solution of this problem consisted in the investigations of the expressions which represent $F(x)$ not only within the circle of convergence C of $p(x)$ but also on the circumference of C at those points at which $F(x)$ is regular. Borel is the first who arrived at a solution more general, in having obtained an expression valid within a domain B which surrounds, in general, the domain C. Borel's idea that he had obtained, by his summation expression, the power series itself in the case where it diverges, was characterized as a play upon words, all the more intemperate since it gives rise to the illusion, entirely false, that he has been able to extend the limits of the theory of analytic functions beyond those fixed by the classic theory. As one is able, since the expression of Borel is convergent, to perform the same operation on the divergent series as on the convergent series $p(x)$, this scheme implies only the translation, for this special case, of the theorem of Weierstrass just stated.

The complete solution of the problem of the lecture has been obtained finally and in several different ways since the new conception of "the star" (of convergence) was first introduced by Mittag-Leffler in 1898, and one of the solutions, which was obtained by the generalizations of the integrals of Laplace, is attached to the important study of the functions $E_a(x)$ of Mittag-Leffler, as well as to that of the increase of the integral functions in an angle or along different semi-lines.

The lecturer closed by recalling the remarkable property of the function

$$E(x)=\int_{S} e^{e^{z}}\cdot e^{-e^{z}}\,\frac{dz}{z-x},$$

which tends indefinitely and uniformly toward zero when the variable x increases above any limit in the interior of a domain surrounding an angle, however small it may be, and which embraces the infinite part of the real positive axis, but which has the unexpected property of tending uniformly toward zero when the variable increases toward infinity along the real positive axis.

After a short recess Professor Forsyth read his address: "On the present condition of partial differential equations as regards formal integration."

The aim of the lecturer was to call attention to some of what may be called the classic methods of the past, whether recent or more remote, stating that in his opinion their usefulness had not been exhausted. He pointed out the need of a systematic treatment of the subject, since the theory remains fragmentary in form, indeed so much so that each individual success toward the solution of the problem is built up almost independently of its predecessors. The lecture was limited to the consideration of equations of the second order containing one dependent and two independent variables.

Two definitions of general integral were discussed, viz., that of Ampère and that of Darboux. The opinion was expressed that, until the old methods of integration are extended or new and more comprehensive methods given, we are in no position to declare what a comprehensive integral is. The lecturer then proceeded to what he considered the three principal methods of constructing the integral of a partial differential equation of the second order. These methods are the following:

(1) The method of Laplace: It is applicable to strictly linear equations and is effective for them solely when there are two independent variables, but the extension to equations of higher order still remains to be made.

(2) The method of Ampère: The results achieved by Ampère, particularly when regard is paid to the date, were on a

grand scale. Even now his ideas have not received their full development and his discriminating classification remains unamplified to this day. Unfortunately his method, while general in form and spirit, has no compelling force ; success in actual practice depends upon individual skill. His first class of equations, consisting of those whose integrals are expressible in finite terms without partial quadratures, remains as he defined it.

(3) The method of Darboux : When stated in full it can be effectually applied to equations which possess an intermediate integral. The general idea is based on the construction of another equation (still better two other equations) of the same order as the given equation and compatible with it. Moreover the method is progressive, that is to say, when the analysis shows that no compatible equations of the same order exist, similar analysis may lead to the construction of compatible equations of higher order.

One important gap still remains to be filled in this last method. It is true that when the equation is of the required character, the primitive can be constructed by this method, but no method has yet been devised for showing whether a given equation is of the required character. What is needed to fill the gap is one of two things, either some test which will show a priori the lowest order of compatible equations that can be associated with a given equation or else the construction of all the partial equations amenable to the method.

It happens only too often that an equation is not amenable to any one of the methods which have been mentioned. In that case the only plan of deriving information concerning the primitive is to have recourse to Cauchy's existence theorem. Usually this presents itself in inconvenient form, but at present we can do no better.

Some suggestive results have been obtained by proceeding from a different initial standpoint, using once more such classic processes as those of Euler and Lagrange. The principle of such methods is to regard the primitive as the origin and the partial equation as the consequence of the relation between the primitive and the equation.

We have thus seen that in this subject we can take only a few steps before reaching the boundary of present knowledge.

On Tuesday morning the four sections met for the first time, organized, and proceeded to the reading of special papers.

Each morning for the remainder of the week was devoted to the sectional meetings. On Tuesday afternoon two general papers were read. The first was by Professor Darboux on "Les méthodes et les problèmes de la géométrie infinitésimale."

The lecturer first gave a short account of the origin of infinitesimal geometry. Like other branches of human knowledge it had its origin in a practical problem, viz., that of mapping the surface of the earth. Lambert was the first to propose the problem, in full generality, of mapping the surface of the earth on a plane in such a way that infinitesimal elements shall be similar. It gave rise to some beautiful researches of Lambert, Euler, and Lagrange, but the complete solution was first given by Gauss in 1822. This was followed in 1827 by his Disquisitiones. In this last are found many ideas destined to great development in the modern theory of surfaces.

Professor Darboux next spoke of the activity in France during the same period and made special mention of the works of Monge, Dupin, Chasles, and Lamé, and the influence which they had on the development of infinitesimal geometry. Mention was also made of the geometers who followed these, and Ossian Bonnet was characterized as one to whom his works give almost the role of creator.

Having thus briefly sketched the history of the subject, the lecturer proceeded to what he considered the proper method of infinitesimal geometry. He said the method should be that of analysis which makes use of coordinate axes. The research should always be vivified and inspired by the geometric spirit. The method should not be followed blindly, for this being a grand highway will lead the more securely to the end desired, but the other roads of travel indeed have their charm. He advised the complete and frank introduction of imaginaries.

He then passed to the discussion of some of the unsolved problems in infinitesimal geometry.

1. The curves of constant torsion are of great importance in the study of surfaces applicable to a paraboloid of revolution, as these surfaces can be derived by a geometric construction from imaginary curves of constant torsion. Now it would be a problem worth the effort to determine all those curves of constant torsion which are algebraic or even unicursal. Attempts have been made at the problem, but a general solution is yet to come.

2. Bour announced that by applying the celebrated method of

Lagrange, the variation of constants, he had obtained all the surfaces applicable to any surface of revolution whatever, and in particular to a sphere. The manuscript was lost in a fire. Many geometers have sought to find the solution of Bour, but as yet no one has succeeded.

3. The initial conditions selected in the discussion of the problem of Cauchy should be as general as possible. For example, the following are suggested : If it is a question of an equation with two independent variables, the surface should be restricted to pass through a given curve when the equation is of the first order, but if the equation is of the second order the surface should be tangent the whole length of a curve to a developable surface. This method should lead in a precise manner to the notion of characteristics.

4. The above problem has been solved completely for minimal surfaces, but it is not to be confounded with the very different problem which has for its object the determination of a continuous minimal surface which passes through a given closed curve. Much has been done, but much still remains to be done with this latter problem.

5. Along with the problems of mapping, the problem of Tchebychef, somewhat generalized, was considered. Imagine a net, of any form and dimensions, composed of two series of threads attached firmly at their points of intersection so that the angles but not the sides of the meshes may vary. Suppose a solid bounded by any surface whatever be dropped into the net. Determine the form which the net assumes. In case the solid is spherical the solution can be connected with the determination of surfaces of constant curvature.

6. Some of the first and most elementary problems connected with the quadratic differential forms are yet to be solved. Here the great discoveries of Lie on the theory of groups have found and will continue to find beautiful applications.

7. The problem of the deformation of surfaces has been solved for the paraboloid, quadrics of revolution, and quadrics tangent once to the circle at infinity. Can the results be extended to quadrics in general? The answer, whether positive or negative, will be interesting.

The lecture closed with showing the important part which infinitesimal geometry plays in the study of partial differential equations.

Following Professor Darboux, Professor von Dyck presented a report: "Über die mathematische Encyklopädie."

On Wednesday afternoon Professor Newcomb read his address: "La théorie du movement de la lune: son histoire et son état actuel."

Since the law of gravitation was enunciated it has been the work of many mathematicians to compute theoretically the movement of planets, and in nearly all cases the theoretic movement agrees with the observed movement, but there are two noted exceptions, viz., Mercury and the moon. In the case of Mercury the difference can be accounted for by the presence of an unknown body between Mercury and the sun, but in the case of the moon we are confronted by an enigma. The importance of this divergence can be recognized after we have sketched the present state of the theory of the movement of the moon.

The law of gravitation is expressed by means of three differential equations, with time as the independent variable. The deductions are obtained by integrating these equations. The integrals give the coordinates of the body in terms of t and six arbitrary constants. The general solution in the case of the moon is not possible, but on account of the smallness of certain elements we can obtain a solution in the form of an infinite series arranged in powers and products of these elements. The coefficients are periodic functions of the time and we have solved the problem when we have calculated these. The method of Delaunay for handling these equations was discussed. He reduces the solution to the execution of a series of algebraic operations, comprising substitutions repeated without end but always approaching more and more nearly the exact value of the variables. This series for the coefficients, however, is so slowly convergent that it is not sufficient for the needs of modern astronomy.

There is a method used by Hansen which consists in substituting the numerical values for the constants, but since the derivatives cannot be calculated we must consider this method as unsatisfactory.

However there is a method which, in the speaker's opinion, has all the exactness demanded by the astronomy of our time. This is the method initiated by Euler and completed by G. W. Hill nearly a century later. Considering the mean movement of the moon about the earth and of the earth about the sun as known, the coordinates can be expressed in a series arranged according to powers of e, e' (the eccentricities of the two orbits), and $\gamma = \sin \frac{1}{2}I$. Then, y being such a coordinate,

$$y = P_0 + eP_1 + e'P_2 + \gamma P_3 + e\gamma P_4 + \cdots,$$

where the P's have the form

$$P = \sum {}^{\sin}_{\cos} (A + Bt).$$

Euler sought to develop separately the values of P, but found difficulty in introducing e, e', γ in B. Hill succeeded in developing P_0 in terms of the mean movement. He also gave a method for treating that part of the motion in perigee, independent of e, e', γ. E. W. Brown has attacked the problem also and has conquered its difficulties. The degree of exactness of Brown's work is all that could be desired.

These investigations all assumed that the sun is the only source of perturbation, but the planets also exert a small force which causes an acceleration of the mean movement of the moon, and there is besides a disagreement between theory and observation of which William Ferrel has found the partial cause in the effect of the moon upon the tides. But even this does not account for the entire disagreement. To explain the variations in the mean movement of the moon by the action of the tides, it would be necessary to suppose variations of almost a minute in our measure of time during the last two centuries. But the transits of Mercury seem to show that the variations cannot be more than a few seconds.

The lecturer closed by saying that the enigma seemed to him one of the most important and most interesting problems in celestial mechanics.

After a short recess Professor Lorentz gave his address: "Le partage de l'energie entre la matière ponderable et l'éther."

Kirchhoff reached the conclusion that the energy of radiation which exists in a unit volume of ether, so far as it depends upon a wave length comprised between the limits λ and $\lambda + d\lambda$, can be represented by an expression of the form

$$F(\lambda, \tau)d\lambda,$$

where F is independent of the special properties of the body. The lecture was especially concerned with the form of the function F. Wien has given to the function the form

$$F(\lambda, \tau) = \frac{1}{\lambda^5} f(\lambda\tau),$$

where instead of two independent variables λ, τ we have only the product $\lambda\tau$, but the form of f is still undetermined.

Professor Lorentz pointed out that Maxwell and Boltzmann had made use of two important branches of mathematics, viz., the calculus of probability and geometry of n dimensions. After showing the importance of these, he proceeded to limit the problem to considering a small mass M, composed of innumerable atoms animated and in motion in a rectangular box; further there are some charged particles or electrons free or held captive in the interior of the atoms. These electrons take part in the calorific movement of the atoms and should be regarded as the source of radiation.

The method used to treat the problem was that due to Gibbs. The state of the ether when some movable electrons are present is expressed by a system of partial differential equations seemingly very different from those of Hamilton, but which can be reduced to those by first establishing a theorem analogous to that of least action.

It is found that one meets a difficulty in applying the Gibbs method because the number of coordinates which define the electric field in the ether is infinite. It is necessary, therefore, to replace this system by a virtual system which has n degrees of freedom and treat the real system as the limit of this as n becomes infinite.

Finally the form of the function of radiation is found to be

$$F(\lambda, \tau) = \frac{16\pi a\tau}{\lambda^4}.$$

This, however, is only valid for great wave lengths, but is definite enough for these. α is a universal constant which can be computed by making measurements on the infra-red rays. Planck by an entirely different method has obtained the formula

$$F(\lambda, \tau) = \frac{8\pi ch}{\lambda^5} \cdot \frac{1}{e^{3ch/2a\lambda\tau} - 1}$$

which for long wave lengths agrees with the previous result.

The fact that F is independent of the special properties of the body is accounted for by the energy of agitation of the constituents, represented by $a\tau$, which determines the intensity of radiation in the ether.

The lecturer closed by considering the merits and defects of the various methods for determining *F*.

On Wednesday evening a reception was given in the museum of sculptors at the Capitol, by the municipality of Rome. The halls were brilliantly lighted and beautifully decorated and their capacity, though very large, was tested. Mayor Nathan and other officials of the city were present. This reception was no less a delight to the members than the one on Sunday evening and the halls were not cleared until after midnight.

The usual program was followed Thursday morning, but in the afternoon the general addresses were suspended, and the congress adjourned to the Palatine, where a most delightful season was spent in viewing the many things of interest contained there. Guides who spoke different languages were furnished. After spending some time in inspecting the old Roman remains we were conducted to a beautiful spot on the Hill where a fine lunch was served. Everyone went away with the feeling that these non-sessions were among the most delightful parts of the congress.

Thursday evening all were provided with tickets to the concert at the Amphitheatre Corea. Those who loved music certainly enjoyed a treat and the others did not regret going.

Friday afternoon, Professor Poincaré being ill, his address was read by Professor Darboux. The title was: "L'avenir des mathématiques."

The author first examined the tendencies of the past and the present in order to be able to predict the future from these. He finds that there are two forces which influence mathematical thought. The first is the demand which physicists and engineers make on the mathematician for the solution of their problems. The second is the demand for the economy of thought. This economy is effected by organization and in many other ways; even the elegance to which mathematicians attach so much importance is attributed to this economy of thought.

In this economy it is not sufficient to give models to be followed. It is necessary that once knowing a given piece of reasoning, it can be repeated, in substance, in a few words. That is to say, a special choice of words is necessary. This choice of words has also another merit, it enables one to see that demonstrations made concerning one class of objects are valid for another also. Two good examples of this are the words group and invariant.

Having determined that these are the forces which influence mathematical thought to-day, the author proceeded to make predictions concerning various branches of mathematics.

Arithmetic: The development of arithmetic is now much behind that of algebra and analysis, mainly because of the discontinuity of its elements. The future of arithmetic therefore lies in the direction of making use of the advances in algebra and analysis in order to develop itself. For example, the parallelism between algebraic equations and congruences is almost perfect. Certainly this can be carried to completion.

The theory of primes in arithmetic seems to lack unity almost entirely, but without doubt unity will be produced by the consideration of a family of transcendental functions which permit, by the study of their singular points and the application of the methods of Darboux, the calculation asymptotically of certain functions in very great number.

Algebra: In the study of algebraic equations there still remains the problem of a system of invariants which do not change sign when the number of roots remains the same.

If one forms power series representing functions which admit the roots of an algebraic equation for singular points, the coefficients of the terms of higher order will furnish one of the roots with an approximation more or less close. Here is the germ of a process for numerical calculation of which a systematic study could be made.

The theorem of Gordan in the theory of invariants is yet to be extended.

Differential equations: In studying the functions defined by a differential equation we will not be satisfied until we have found groups of transformations which play the same rôle with respect to the differential equation that the groups of birational transformations play with respect to algebraic curves.

The lecturer emphasized the qualitative discussion of the curves defined by a differential equation. This is analogous to the investigation of the number of roots of an algebraic equation.

Partial differential equations: Here the idea of Fredholm can be applied to all the equations of the linear form. This remains to be carried to its completion.

Theory of functions: Here the theory of functions of more than one variable will occupy the attention; a mere analogy to functions of one variable is not sufficient. We must search for the things which will throw light on the difference between the theory for one variable and for several. For example, what is to take the place of conformal representation?

Theory of groups: The theory of continuous groups being much more advanced than the theory of Galois groups, it remains for the latter to make use of the analogies with the former.

Geometry: Analysis situs has been of great aid to mathematicians and still awaits to be completely constructed for space of more than three dimensions. When this is done we shall have an instrument which shall permit us really to see in hyperspace. The study of groups in geometry has contributed much to its growth. The study of groups of curves on a surface, analogous to the groups of points on a curve studied by Brill and Noether, has never been worked out.

Postulates: It seems at first that this field is well limited and that there is nothing to do when the inventory is completed. But when all are enumerated, there will be different ways of classifying them and each new classification will be of interest to the philosopher.

After a short recess Professor Picard gave his address: "L'analyse dans ses rapports avec la physique mathématique."

The principal object of the lecture was to show the mutual relation between physics and mathematics, that is to show that each derived much benefit from the other.

In the seventeenth century the development of kinetics and dynamics gave birth to the greatest advances in analysis; from that time dates the real beginning of modern analysis, it really came from mechanics. The origin of the notion of derivative is involved in the idea of the movability of things and of the rapidity with which phenomena occur. It was a decisive epoch in the history of mathematics when the development of mechanics conducted to the postulation that infinitesimal changes depend uniquely on the actual state of the system.

In the eighteenth century the development of mathematics in its most essential points was identified with that of mechanics. Clairaut in his study of the form of the earth considered, for the first time, curvilinear coordinates, d'Alembert used the two fundamental equations of the theory of functions and saw the significance of $\sqrt{-1}$. We find also functions of a complex variable considered by Euler and Lagrange.

The theory of potential and the analytic theory of heat have contributed much to mathematics. In these we often find also suggestions of the method of solution of the mathematical problem. Thus in this case physics has rendered a double service to mathematics.

There are also many examples of the service which mathematics has rendered to physics. The monumental work of Green on the applications of analysis to electricity and magnetism is an excellent example. In the study of the theory of waves we find also that differential equations have rendered much service; for example, in the theory of heat they showed that any variation is felt instantly in all directions and that therefore we cannot speak of the velocity of propagation. The influence which formal mathematics has had on physics and mechanics are instanced also by the equations of Laplace and the principle of least action.

The knowledge of the integrals of partial differential equations has caused some noted paradoxes in physics to vanish. The modern theory of functions seems at first to present but little of interest to the physicist, but extended study shows the theory of analytic functions to be of extreme importance to the mathematical physicist. The questions of domain of existence and of the prolongation of integrals have no less interest for the physicist than for the analyst. Indeed it seems to us that analysis is an indispensable instrument to the physicist and in some cases even a precious guide.

On Friday evening Professor Störmer gave his address: "Sur les trajectoires des corpuscles électrisés dans le champ d'un aimant élémentaire, avec application aux aurores boréales." The lecture was illustrated by a set of lantern slides. It was a résumé of a large memoir on the same subject, published in 1907 in the *Archives* of Geneva.

Partly by analysis and partly by the powerful methods of numerical and graphic integration of differential equations, Professor Störmer has succeeded in finding the properties of the trajectories in question. The numerical calculation represents a labor of nearly 5,000 hours. The author showed afterwards how the results found sufficed to explain a whole series of details in the remarkable experiments of M. Birkeland, in which he exposed a magnetic globe to a pencil of cathode rays, as explaining the principal character of the aurora borealis, for example the zones of auroras, the appearance in the night, the auroral rays and the remarkable phenomena of draperies of auroras.

The results of the applied analysis now make very probable the hypothesis of M. Birkeland, according to which the auroras are due to cathode rays, or something analogous, emitted by sun spots and meeting the atmosphere of the earth under the action of terrestrial magnetism.

On Saturday afternoon, Professor Veronese being ill, his lecture on "Geometria non-archimedèa" was not read, but it will appear in full in the Atti del Congresso. The congress proceeded to the consideration of recommendations made by the various sections. The following resolutions were adopted :

The congress, having recognized the importance of an accurate examination of the program and the methods of instruction in mathematics in the secondary schools of the various nations, appoints Professors Klein, Greenhill, and Fehr as an international committee to study the question and to report upon it at the next congress.

"Section III (mechanics), after an exchange of views in which the importance of a unification of the notation of vectors was recognized, proposes to the congress the nomination of an international committee for the study of this question. The president of this section for the session of April 11 proposes to the congress to adopt its committee of organization to constitute this commission, and submits the list of names."

"The congress votes that at the next congress the constitution of the International Association of Mathematicians be presented."

"It is clear, from the exchange of views in Section III-b, that it would be highly desirable to effect a closer union between those who are occupied in perfecting mathematical methods and those who make practical applications. To this end the section urges that mathematics applied to the science of engineering be the object of a special session at the next congress. Section III-b further proposes the appointment of an international commission, which shall have charge of the work of this new section. The composition of this international commission shall be fixed by the bureau of the fourth congress."

"The Fourth International Congress of Mathematicians in Rome considers as a matter of maximum importance to the mathematical sciences, both pure and applied, the publication of the complete works of Euler. The congress greets with approval the initiative taken by the society of Swiss naturalists, and votes that the great work be taken up by the society itself with the help of the mathematicians of other nations. The congress urges the international association of the academies, and especially the Academies of Berlin and St. Petersburg, of which Euler was a celebrated member, to aid the enterprise."

The congress unanimously accepted the invitation of the Cambridge Philosophical Society to hold its fifth meeting at Cambridge in 1912.

Professor Mittag-Leffler, on the part of the Swedish mathematicians and the King of Sweden, invited the Congress to hold its sixth meeting at Stockholm in 1916. This of course could not be voted upon at this meeting.

Professor Darboux in the name of all the members of the congress thanked the committee and all those who had had a part in making the fourth international congress so important and so pleasant. The president then closed the congress officially.

While the official end of the congress was Saturday, there was still one more very pleasant non-session to be held on Sunday. All were furnished with tickets to Hadrian's Villa and Tivoli. The first stop was at Hadrian's Villa. Carriages were ready to take those who did not care to walk from the station to the villa. On entering the ruins, we found refreshments, provided by the municipality of Tivoli, ready and waiting to be served. After spending about two hours here, we proceeded to Tivoli, where a banquet awaited us. The banquet closed with toasts in Italian, French, German and Latin.

The afternoon was spent in visiting the cascades and the Villa d'Este. The returning trains arrived in Rome about 8 P. M. This was the unofficial but real close of the congress.*

C. L. E. MOORE.

ROME, *May*, 1908.

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* A report of the sectional meetings of the Congress will appear in the October BULLETIN.

SHORTER NOTICES.

Encyklopädie der Elementar-Mathematik. Ein Handbuch für Lehrer und Studierende. Von HEINRICH WEBER und JOSEF WELLSTEIN. Zweiter Band, *Elementare Geometrie.* Bearbeitet von HEINRICH WEBER, JOSEF WELLSTEIN, und WALTHER JACOBSTHAL. Leipzig, Teubner, 1905. 8vo. 280 figures. viii + 602 pp. Dritter Band, *Angewandte Elementar-Mathematik.* Bearbeitet von HEINRICH WEBER, JOSEF WELLSTEIN, und RUDOLF H. WEBER. Leipzig, Teubner, 1907. 8vo. 358 figures. ix + 666 pp.

EVERY live teacher of secondary school mathematics is aware of the superficial character of most text-books. In the nature of the case, teaching for younger pupils must exclude far more than it presents. But if the teacher himself restricts his study to the range prescribed for pupils, very little mathematical interest is kindled in his classes. Practically the same result is reached if the sole scientific interest of the teacher is in fields remote from his pupils' studies. The authors of this three-volume encyclopedia of elementary mathematics plan to intensify by fundamental criticism, and revivify by extensive applications to questions of physics, the interest of the young teacher in his every-day work. This does not conflict by any means with the programme of modern universities — to train the future teacher by research in some region on the frontiers of scientific knowledge. Rather it supplements that programme, and strengthens the position of its champions, by showing how to apply the method of the university seminar to the problems of the schoolroom.

The book on geometry begins with a critical and historical survey of the notions point, straight line, surface, plane, parallel. The antithesis is developed between what the authors call natural and logical geometry, the latter reached only by a limit process of idealization. Most appropriate is then the quite full examination of a second system of geometrical objects, the totality of spherical surfaces that contain a fixed point O; for in this system there is an exact correspondence to the objects of ordinary euclidean geometry, while the images are radically different. Straight lines are replaced by circles through O, and planes by spheres through O. Tried upon this material, the Hilbert axioms of connection and of order lose their appearance of artificiality and become novel observations of fact. Then an exchange (duality) of sphere of spheres for line, etc., serves to shake loose the possession that point must mean always one thing, truth of the theorems of Euclid's geometry is mystic dent upon something more than the axioms. Soon is i also the three-fold infinity of circles in a plane: each called a point, each pencil of circles a line, and each "B a plane. It is easy thence to foresee how there must elliptic and hyperbolic non-euclidean geometries in a plan on a spherical surface.

By easy stages the reader is led to Hilbert's inquiries respecting the independence of axioms, and to the "pathological" systems constructed for the purpose of establishing that independence. The climax of this fascinating chapter is reached in the paragraph 14, where metageometry is considered in its relation to philosophy, especially to Kant. It will interest the casual reader to note (page 144) the explicit opinion: "Necessities of intuition there are none; necessity can lie only in acts of the intellect;" while a footnote states that on this point there is a difference of opinion between the two editors.

Projective geometry is treated no less carefully as to its axioms, conics are considered in their main features — and shown to be central projections of circles, and projective metric concludes this third section, to which is added a well chosen list of references.

Planimetry (pages 220–301), shows the same influence of Hilbert's "Foundations." In particular it contains a section that will be welcome to very many teachers of geometry, upon the number π and the history of circle measurement. Inversion with respect to a circle, and the problem of Apollonius appear as extensions of the traditional school-book material.

Trigonometry, plane and spherical, is given a concise discussion. New to many will be the division of formulas into those of the first order and those of the second, and the generalization of the spherical triangle (Gauss and Study), with Study's theorem that the totalities of proper and of improper spherical triangles form two discrete continua, with no continuous transition from the one to the other by moving the vertices freely in the surface of the sphere. Three points of the surface denote a great number of different triangles when sides and angles are unrestricted in magnitude, and for both the distinction is maintained between positive and negative. Adopting any modulus, as 2π, 4π, 6π, etc., renders the number of non-equivalent classes of triangles with three fixed vertices finite, and Study's theorem is valid for all moduli.

Brief but artistic sections on analytic geometry of the plane (76 pages) and of space (72 pages) include much of value, as the elements of integration for volume, and the rotation groups of regular solids. A good index, and a full supply of clear diagrams, make this a valuable book of reference even for teachers who will read it but infrequently.

Of the third volume it is not too much to say that it contains a most valuable presentation of physical theories for mathematical teaching. That it is kept free from overloading of theory is seen perhaps in the fact that continuity and discontinuity are not mentioned in the index, nor critical states of matter. Half the volume is physics, vector geometry, analytical statics, dynamics, electricity and magnetism, and electromagnetism. Of the remainder, maxima and minima in geometry and capillarity fill 43 pages; probability and least squares, 40 pages; and a full and suggestive book on graphical statics the concluding 240 pages.

In a note appended to this volume, H. Weber reverts to the Mengenlehre of the first volume, cites Russell's paradox on the class of classes that do not contain themselves (which he identifies with one of Kant's antinomies); and gives an outline discussion of finite aggregates, free from objections, as he believes. These volumes certainly constitute a valuable work for every reference library.

H. S. WHITE.

Leçons sur l'Intégration et la Recherche des Fonctions Primitives. Par HENRI LEBESGUE. Paris, Gauthier-Villars, 1904. 8vo. viii + 138 pp.

SINCE the publication of Lebesgue's thesis in 1902 the originality and power of his methods have attracted increasing attention to the field in which he and Baire have made such important contributions. They have given to the study of discontinuous functions an impulse which is apparent on the most cursory survey of current mathematical periodicals, and of such recent treatises as those of Young and Hobson.

The present volume, one of the series of monographs published under the direction of Borel, reproduces a course of twenty lectures delivered at the Collège de France on the Peccot foundation. Within such limits one could hardly expect a complete treatment of so large a subject as the title suggests, and in fact the author restricts himself to real functions of a single real variable. The only applications considered are connected with the problem of primitive functions and the rectification of curves, the author's researches on trigonometric series being left for a subsequent volume of the same collection. Even with this narrowing of the field the material seems almost too abundant and the reader arrives at the end of each chapter almost out of breath from the rapid pace which the author sets. One is inclined to wonder whether this haste of treatment did not extend to the editing of the volume. There is an unusual number of misprints and inaccuracies in formulas, particularly in the last chapter, and in three or four instances rather extended proofs are faulty. Lebesgue has himself recently indicated some important corrections in the *Atti della Reale Accademia dei Lincei* (series 5, classe di scienze fisiche, etc., volume 15, 1906). It should be added however, that the theorems concerned are true as stated; the inaccuracies are in the proofs. In some places brevity interferes with clearness, particularly in statements where conditions must be supplied by the reader. But, apart from these minor defects, one cannot but feel that this is a brilliant piece of work, full of originality, ingenious in its methods, important in its results.

To a certain extent the historical development of the subject is followed, beginning with definitions of integration deduced from the work of Cauchy and Dirichlet and terminating with an exposition of the method developed by the author. Everywhere the theory of point sets is of fundamental importance, and in particular the notion of the measure of a set of points. By a happy extension and simplification of Borel's definition Lebesgue gives to this term a meaning which is especially adapted to functional operations involving an enumerable infinity of elements. This may be briefly indicated as follows for a linear set E belonging to a segment AB: The points of E are enclosed in a set of intervals whose number is finite or enumerably infinite;* if we consider the sum, or the limit of the sum, of the lengths of these intervals it is obvious that for all possible sets enclosing E this sum has a lower limit $m_e(E)$; this is called the exterior measure of E. The interior measure

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* If we omit the words "or enumerably infinite" this becomes the definition of measure in Jordan's sense. Lebesgue's measure includes both Jordan's and Borel's.

$m_i(E)$ is defined as the difference between the length AB and the exterior measure of the set of all points of AB which do not belong to E. If the exterior and interior measures are equal the set is said to be measurable, its measure being the common value of $m_e(E)$ and $m_i(E)$. The extension to sets in more than one dimension is obvious. The generality of this definition may be inferred from the fact that no non-measurable sets have yet been constructed by processes which do not involve such "idealistic" notions as are involved in an arbitrary choice among an infinite number of elements, though existence proofs of the latter kind have been given recently by Lebesgue, Vitali, and Van Vleck. The first-named has, however, constructed sets that are not measurable in Borel's sense.

A first application of the idea thus introduced is to be found in the formulation of the condition for the existence of an integral in Riemann's sense which requires, in the case of a limited* function, that its points of discontinuity in the integration interval form a set of measure zero. But the chief importance of this notion for Lebesgue's work arises from the part it plays in his definition of the integral, a definition which applies to all limited measurable functions, *i. e.*, functions such that for any two numbers α, β the points on the x-axis in the interval $[a, b]$ for which $\alpha < f(x) < \beta$ form a measurable set. The generality of this category of functions is apparent from what is said above as to the existence of non-measurable sets. With minor changes of notation Lebesgue's definition may be thus given: Divide the interval of variation of $f(x)$ into n subintervals by means of the numbers

$$y_0 < y_1 < y_2 < \cdots < y_n = Y,$$

and designate by $m(E_i)$ the measure of the set of points on the segment $[a, b]$ of the x-axis for which $y_{i-1} < f(x) < y_i$, and by $m(E'_i)$ the measure of the set for which $f(x) = y_i$. Then the common limit as n becomes infinite of the two sums

$$S = \sum_{i=1}^{i=n} y_i m(E_i) + \sum_{i=0}^{i=n} y_i m(E'_i), \quad s = \sum_{i=1}^{i=n} y_{i-1} m(E_i) + \sum_{i=0}^{i=n} y_i m(E'_i)$$

is the integral in Lebesgue's sense of $f(x)$ from a to b.

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* Lebesgue uses the term limited for functions all of whose values in the interval considered lie between two constants A and B; a function is finite if it has a finite value at every point of the interval.

If $f(x)$ is integrable in Riemann's sense then this new definition of the integral coincides with the ordinary one. Perhaps the relation of these two ideas is most clearly shown by means of the geometric conception of the integral as the difference between the two-dimensional measure of the set of points in $[a, b]$ having positive ordinates for which $0 < y < f(x)$, and that of the set of points for which these signs of inequality must be reversed. If these measures exist according to Jordan's definition, their difference is the integral in Riemann's sense; the insertion of the words "or enumerably infinite" in the right place in the definition of measure gives us the Lebesgue integral.

In so brief a notice as this it is impossible even to indicate the many applications which the author makes of this new and powerful instrument of analysis. There still remain unsolved cases of the problem of the primitive function, *i. e.*, the problem of determining a function whose derivative is given, but the solution is found in many cases where the ordinary integral cannot be used. In particular the question is settled whenever the given function is limited, or when it is known that the primitive function must be of limited variation. And in the case of rectifiable curves the Lebesgue integral gives their length whenever the functions $x(t)$, $y(t)$, $z(t)$ which define the curve have limited derivative numbers.

The treatment of several topics connected with the main subject should be mentioned, especially the chapters and sections on functions of limited variation and on derivative numbers. The volume closes with an admirably clear and concise note on point sets and transfinite numbers.

D. R. CURTISS.

NOTES.

AT the meeting of the London mathematical society held on May 14 the following papers were read: By P. A. MACMAHON, "On the invariants of the general linear homogeneous transformation in two variables"; by H. HILTON, "On the order of the group of isomorphisms of an abelian group."

A NEW academy of sciences has been established in Finnland, with seat at Helsingfors. Two sections have been organized, one consisting of mathematics and the physical sciences, the other consisting of philology and philosophy.

THE French association for the advancement of science will hold its annual meeting at Clermont Ferrand during the first week in August.

THE following advanced courses in mathematics are offered during the academic year 1908–1909 :

JOHNS HOPKINS UNIVERSITY. — By Professor F. MORLEY : Higher geometry, two hours ; Dynamics, two hours (first half year) ; Theory of functions, two hours (second half year) ; Reading course, one hour ; Seminary, one hour. — By Dr. A. COHEN ; Introduction to the theory of functions, two hours ; Differential equations and differential geometry, two hours. — By Dr. A. B. COBLE : Theory of groups, two hours.

UNIVERSITY OF PENNSYLVANIA. — By Professor E. S. CRAWLEY : Modern analytic geometry, two hours ; Theory of numbers, three hours. — By Professor G. E. FISHER : Differential equations, two hours ; Elliptic functions, three hours (first half year) ; Topics in the theory of differential equations, three hours (second half year). — By Professor I. J. SCHWATT : Infinite series and products, two hours. — By Professor G. H. HALLETT : Lie's theory of continuous groups, three hours (first half year) ; Galois's theory of equations, three hours (second half year) ; Groups of finite order, three hours. — By Professor F. H. SAFFORD ; Curvilinear coordinates, three hours ; Partial differential equations, three hours. — By Dr. O. E. GLENN ; Invariants, three hours.

UNIVERSITY OF BOLOGNA. — By Professor C. ARZELA : Dirichlet's principle, calculus of variations, three hours. — By Professor L. DONATI : Mathematical theory of elasticity, physical optics, three hours. — By Professor S. PINCHERLE : Algebraic functions and their integrals, elliptic functions, analytic functions represented by definite integrals, three hours.

UNIVERSITY OF CATANIA. — By Professor M. DE FRANCHIS : Geometry on algebraic curves, Riemann's surfaces and abelian integrals, inversion, four and a half hours. — By Professor S. LAURICELLA : Optics, four and a half hours. — By Professor G. PENNACCHIETTI : Elliptic functions with applications to mechanics, four and a half hours. — By Professor C. SEVERINI : Application of Lie's theory of continuous groups to differential equations, researches of Picard and Vessiot, four and a half hours.

UNIVERSITY OF GENOA. — By Professor G. FUBINI: Theory of continuous groups and of automorphic functions, three hours. — By Professor G. LORIA: Infinitesimal geometry, three hours. — By Professor O. TEDONE: Maxwell's theory of electromagnetic fields, three hours.

UNIVERSITY OF MESSINA.—By Professor G. BAGNERA: Partial differential equations of the second order, three hours. — By Professor T. BOGGIO: Integral equations with applications to mathematical physics, three hours. — By Professor V. MARTINETTI: Theory of algebraic plane curves and surfaces, curves and surfaces of order 3, three hours.

UNIVERSITY OF NAPLES. — By Professor F. AMODEO: History of mathematics: The XVIII century, three hours. — By Professor A. CAPELLI: Theory of algebraic forms, four and a half hours. — By Professor R. MARCOLONGO: Theory of potential and integral equations, mathematical theory of elasticity, four and a half hours. — By Professor D. MONTESANO: Line geometry, birational transformations of the plane and space, four and a half hours. — By Professor E. PASCAL: Partial differential equations of the second order, three hours. — By Professor L. PINTO: Physical optics, four and a half hours.

UNIVERSITY OF PADUA. — By Professor F. D'ARCAIS: Discontinuous groups of linear transformations, elliptic and modular functions, four and a half hours. — By Professor A. FAVARO: History of optics, especially the invention of the telescope, three hours. — By Professor P. GAZZANIGA: Theory of numbers, three hours. — By Professor T. LEVI-CIVITA: Hydrodynamics, four and a half hours. — By Professor G. RICCI: Mathematical theory of elasticity with applications to optics, four hours. — By Professor F. SEVERI: Theory of continuous groups, two hours; algebraic functions of two variables, two hours. — By Professor G. VERONESE: Synthetic geometry of hyperspaces, four hours.

UNIVERSITY OF PALERMO. — By Professor M. GEBBIA: Mechanics of continua, newtonian attraction, hydrostatics and hydrodynamics, four and a half hours. — By Professor G. B. GUCCIA: General theory of algebraic curves and surfaces, four and a half hours. — By Professor A. VENTURI: Modern views on celestial mechanics, four and a half hours.

UNIVERSITY OF PAVIA. — By Professor E. ALMANSI:

Hydrostatics and hydrodynamics, three hours. — By Professor L. Berzolari: Algebraic curves and surfaces, three hours. — By Professor G. Vivanti: Calculus of variations, three hours.

University of Pisa. — By Professor E. Bertini: Geometry of hyperspace, algebraic geometry with applications, three hours. — By Professor L. Bianchi: Functions of a complex variable, general properties of automorphic functions, four and a half hours. — By Professor U. Dini: Advanced calculus, Bessel functions and integral equations, four and one half hours. — By Professor G. A. Maggi: Theory of electromagnetic phenomena with special regard to the new hypothesis, four and a half hours. — By Professor P. Pizzetti: Principles of spherical astronomy, theory of the figures of planets, three hours.

University of Rome. — By Professor G. Castelnuovo: Algebraic functions of one variable and their integrals, three hours. — By Professor V. Cerruti: Partial differential equations of the first order, three hours. — By Professor L. Orlando: Definite integrals and their application to mathematical physics, three hours. — By Professor V. Volterra: Hydrodynamics, four hours.

University of Turin. — By Professor E. D'Ovidio: Theory of functions of a complex variable, abelian integrals, three hours. — By Professor G. Morera: Newtonian potential and attraction of ellipsoids, figure of equilibrium of a rotating fluid mass, three hours. — By Professor C. Segre: General survey of concepts and methods in modern geometry, three hours. — By Professor C. Somigliana: General theory of elasticity, three hours.

Professor H. Poincaré has retired from the professorship of astronomy at the Ecole polytechnique with the title of honorary professor.

At the University of Toulouse, Professor E. Cosserat has, at his own request, been transferred to the department of astronomy. Professor J. Drach, of the University of Poitiers, has been appointed his successor as professor of mathematics.

Dr. L. Bricard, of the Ecole polytechnique, has been appointed professor of mathematics in the conservatory of arts and measures, of Paris, as successor to Professor Lausedat.

DR. G. SANNIA has been appointed docent in geometry at the University of Turin.

DR. G. SCORZA has been appointed docent in projective geometry at the University of Bologna.

PROFESSOR M. ABRAHAM, of the University of Göttingen, has been appointed professor of mathematical physics at the University of Illinois.

DR. A. E. YOUNG, of Purdue University, has been appointed professor of mathematics at Miami College, Oxford, Ohio.

PROFESSOR C. O. GUNTHER, of Stevens Institute, has been promoted to a full professorship of mathematics at the same institution.

MR. E. H. COMSTOCK has been appointed assistant professor of mathematics at the University of Minnesota.

DR. J. H. MCDONALD, of the University of California, has been promoted to an assistant professorship of mathematics.

PROFESSOR O. BOLZA, of the University of Chicago, has been granted leave of absence for the next academic year, which he will spend in Europe.

PROFESSOR J. W. YOUNG, of Princeton University, has been appointed assistant professor of mathematics at the University of Illinois.

AT Princeton University, Drs. C. E. STROMQUIST, J. G. HUN, and C. R. MACINNES have been promoted to assistant professorships of mathematics. Dr. JOHN IRWIN has been appointed instructor in mathematics.

DR. C. N. MOORE has been appointed instructor in mathematics at the University of Cincinnati.

AT the University of Wisconsin, Mr. R. T. CRAIGS has been appointed instructor in mathematics in the university extension department.

MR. W. E. MACDONALD, of the Massachusetts Institute of Technology, has been appointed instructor in mathematics at Harvard University.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Campos y Campo (J.). Estudio sobre la integracion de las ecuaciones diferenciales lineales. Madrid, 1907. 4to. 79 pp.

Cauchy (A.). Oeuvres complètes, publiées sous la direction scientifique de l'Académie des Sciences. (2 séries en 27 volumes.) Série I. Volume 2: Mémoires extraits des mémoires de l'Académie des Sciences. Paris, 1908. 4to. Fr. 25.00

Cicioni (G. C.). La divisibilità dei numeri e la teoria delle decimali periodiche. Perugia, Perugina, 1908. 16mo. 11 + 150 pp. L. 2.50

Elfrink (W. F.). De meetkunde der kegelsneden en eenige harer toepassigen. Harlem, 1907. 8vo. 219 pp. M. 6.50

Encyclopédie des sciences mathématiques pures et appliquées. Edition française publiée sous la direction de J. Molk. Tome I, volume 3: Théorie des nombres. Rédigé dans l'édition allemande sous la direction de F. Meyer. Fascicule 2. Paris, Gauthier-Villars, 1908. 8vo. Pp. 97–192. Fr. 3.00

Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Vol. IV (in 4 Teilbänden): Mechanik, redigiert von F. Klein und C. Müller. 1er Teil, 1te Abteilung, 4tes Heft. Leipzig, Teubner, 1908. 8vo. 16 pp. + pp. 435–691. M. 7.80

—— Vol. VI, 2ter Teil: Astronomie, redigiert von K. Schwarzschild. 2tes Heft. Leipzig, Teubner, 1908. 8vo. Pp. 195–334. M. 4.00

Fulco (P.). La derivata logaritmica considerata come un' operazione. Lodi, Wilmant, 1908. 4to. 12 pp.

Garbieri (G.). Geometria analitica: riassunto di lezioni date nell'università di Genova. Parte I: Luoghi di primo grado, con un cenno su altri luoghi in coordinate cartesiane. Torino, Paravia, 1908. 8vo. 60 pp. L. 3.00

International Catalogue of scientific literature, published by the Royal Society of London. Mathematics. 6th annual issue (1906). London, 1907. 8vo. 126 pp. 15s.

Klein (F.). See Encyklopädie.

Kowalewski (G.). Einführung in die Infinitesimalrechnung, mit einer historischen Uebersicht. Leipzig, 1908. 8vo. 126 pp. M. 1.00

Meyer (F.). See Encyclopédie.

Molk (J.). See Encyclopédie.

Müller (C.). See Encyklopädie.

Neuberg (J.). Cours d'algèbre supérieure. Nouvelle édition, augmentée. Liège, 1907. 8vo. 299 pp. Fr. 6.00

Ollero (D.) y Pérez Griñón (T.). Curso de cálculo infinitesimal. 2a edición aumentada y revisida. Vol. I: Cálculo diferencial. Vol. II: Cálculo integral. Granada, Ventura, 1907. 14 + 343 + 10 + 328 pp. P. 22.00

Pérez Griñón (T.). See Ollero (D.).

Rutgers (J. G.). Einige beschouwingen over de Bessel'sche functies. Alkmaar, 1907. 8vo. 29 pp. M. 1.20

Schwarzschild (K.). See Encyklopädie.

Smith (R. H.). The calculus for engineers and physicists. 2nd edition, revised and enlarged. London, Griffin, 1908. 8vo. 222 pp. Cloth. 8s. 6d

Sylvester (J. J.). Collected mathematical papers. Vol. II: 1854–73, Cambridge, University Press, 1908. 8vo. 747 pp. Cloth. 18s.

Vidal y Rua (A.). Aplicaciones geométricas del cálculo integral á la rectificación de líneas, cuadratura de superficies y cubatura de sólidos. 2a edición. Madrid, "Memorial de Ingenieros del Ejercito," 1907. 115 pp. P. 8.00

II. ELEMENTARY MATHEMATICS.

Adler (A.). Einführung in die Geometrie. Lehr- und Uebungsbuch für die I. Klasse der Realschulen und verwandter Lehranstalten. Wien, Hölder, 1908. 8vo. 3 + 35 pp. M. 0.70

Cannaviello (M.). Corso di geometria elementare per le scuole medie. Parte I: Planimetria. Napoli, Priore, 1908. 8vo. 322 pp. L. 2.00

Crelle (A. L.). Calculating tables giving the products of every two numbers from 1 to 1000 and their application to the multiplication and division of all numbers above 1000. New edition by O. Seeliger. With tables of square numbers and cube numbers from 1 to 1000. Berlin, Reimer, 1908. 7 + 497 pp. Cloth. M. 15.00

Dino (S. N.). Lezioni di trigonometria, ad uso dei licei d'Italia. 11a edizione. Napoli, Trani, 1908. 8vo. 132 pp. L. 2.25

Dorda y López Hermosa (R.). Elementos de cálculo gráfico y monografía y sus aplicaciones prácticas. Texto y atlas. Segovia, Rueda, 1907. 294 pp. P. 12.50

Gallatly (W.). The nine-point circle. With notes on (1) Simson's line; (2) the radical axis; (3) the quadrilateral. London, Simpkin, 1908. 8vo. 1s.

Goyen (P.). Advanced arithmetic and elementary algebra and mensuration. London, Macmillan, 1908. 8vo. 3s. 6d.

Gremigni (M.). Elementi di geometria, ad uso delle scuole tecniche e professionali. Planimetria, libro III e IV. 3a edizione. Firenze, Bemporad, 1908. 16mo. 146 pp. L. 0.90

Hoüel (J.). See Schrön (L.).

Kambly und Langguth. Arithmetik und Algebra. Nach den preussischen Lehrplänen von 1901 umgearbeitet von A. Thaer. Ausgabe A: Für Gymnasien. 39te Auflage der Kamblyschen Arithmetik und Algebra. Breslau, Hirt, 1908. 8vo. 172 pp. M. 2.00

——. Ausgabe B: Für Oberrealschulen, Realgymnasien und Gymnasien mit mathematischem Reformunterricht. 39te Auflage der Kamblyschen Arithmetik und Algebra. Breslau, Hirt, 1908. 8vo. 248 pp. M. 2.50

Langguth. See Kambly.

Mach (E.). Ueber den relativen Bildungswert der philologischen und der mathematisch-naturwissenschaftlichen Unterrichtsfächer. (Schriften des Vereins für Schulreform). Wien, Manz, 1908. 8vo. 32 pp. M. 0.85

Müller (H.). und Witting (A.). Lehrbuch der Mathematik für die oberen Klassen der höheren Lehranstalten. Leipzig, 1908. 8vo. 12 + 320 pp. M. 3.60

Octavio de Toledo (L.). Tratado de trigonometria rectilínea y esférica. Madrid, Viuda é Hijos de Murillo. 285 pp. P. 8.00

Picatoste (F.). Elementos de matematicas. Algebra. Obra declarada de texto para España y Ultramar. Novena edición corregida y aumentada. Madrid, 1908. 208 pp. P. 7.00

Rohrbach (C.). Vierstellige logarithmisch-trigonometrische Tafeln nebst einigen physikalischen und astronomischen Tafeln, für den Gebrauch an höheren Schulen zusammengestellt. 5te Auflage, Gotha, Thienemann, 1908. 8vo. 36 pp. M. 1.00

Schrön (L.). Tables de logarithmes à sept décimales, pour les nombres depuis 1 jusqu' à 108000, et pour les fonctions trigonométriques de dix en dix secondes. Précédés d'une introduction française par J. Hoüel. Edition stéréotype, revue et corrigée. Nouveau tirage. Paris, Gauthier-Villars, 1908. 8vo. 14 + 554 pp. Fr. 10.00

Seeliger (O.). See Crelle (A. H.).

Thaer (A.). See Kambly.

Witting (A.). See Müller (H.).

III. APPLIED MATHEMATICS.

Buckley (R. B.). Facts, figures and formulæ for irrigation engineers. London, Spon, 1908. 8vo. Cloth. 10s. 6d.

Despaux (A.). Explication mécanique des propriétés de la matière. Cohésion, affinité, gravitation, etc. Paris, Alcan, 1908. 8vo. 365 pp. Fr. 6.00

Formulario di matematica e fisica. 7a edizione aumentata. Firenze, Le Monnier, 1908. 10 + 176 pp. L. 1.00

Henriques (P. H.). Föreläsningar i beskrifvande geometri vid Tekniska Högskolan i Stockholm. Häfte 2. Stockholm, 1907. 8vo. 134 pp. M. 2.40

Jamieson (A.). Elementary manual on applied mechanics. 8th edition, revised and enlarged. London, Griffin, 1908. 8vo. 484 pp. Cloth. 3s. 6d.

Sarrette (H.). Précis d'arithmétique des calculs d'emprunts à long terme de valeurs mobilières. Paris, 1908. 8vo. 9 + 287 pp. Fr. 10.00

Schlenscha (J.). See Wildt (J.).

Spooner (H. J.). Machine design, construction and drawing. A textbook for the use of young engineers. London, Longmans, 1906. 8vo 712 pp. Cloth. 10s. 6d.

Wildt (J.). Praktische Beispiele aus der darstellenden Geometrie. Herausgegeben von J. Schlenscha. Lieferung III.; Wien, 1908. 8vo. M. 17.00

SEVENTEENTH ANNUAL LIST OF PAPERS.

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

ALLEN, R. B. On Hypercomplex Number Systems Belonging to an Arbitrary Domain of Rationality. Read Feb. 25, 1905. *Transactions of the American Mathematical Society*, vol. 9, No. 2, pp. 203–218 ; Apr., 1908.

BIRKHOFF, G. D. On the Asymptotic Character of the Solutions of Certain Linear Differential Equations Containing a Parameter. Read (Chicago) Mar. 30, 1907. *Transactions of the American Mathematical Society*, vol. 9, No. 2, pp. 219–231 ; Apr., 1908.

BLISS, G. A. On the Inverse Problem of the Calculus of Variations. Read (Chicago) Apr. 22, 1905. *Annals of Mathematics*, ser. 2, vol. 9, No. 3, pp. 127–140 ; Apr., 1908.

—— A New Form of the Simplest Problem of the Calculus of Variations. Read Apr. 27, 1907. *Transactions of the American Mathematical Society*, vol. 8, No. 3, pp. 405–414 ; July, 1907.

—— A Method of Deriving Euler's Equation in the Calculus of Variations, Read Feb. 29, 1908. *American Mathematical Monthly*, vol. 15, No. 3. pp. 47–54 ; Mar., 1908.

BOLZA, O. On Lagrange's Method of Multipliers in the Calculus of Variations. Read (Chicago) Dec. 28, 1906. *Mathematische Annalen*, vol. 64, No. 3, pp. 370–387 ; Sept., 1907.

—— Existence Proof for a Field of Extremals Tangent to a Given Curve. Read (Chicago) Mar. 30, 1907. *Transactions of the American Mathematical Society*, vol. 8, No. 3, pp. 399–404 ; July, 1907.

BROWN, E. W. On the Lunar Inequality due to the Motion of the Ecliptic and Figure of the Earth. Read Dec. 27, 1907. *Monthly Notices of the Royal Astronomical Society*, vol. 68, No. 6, pp. 450–455 : Apr., 1908.

BUCHANAN, H. E. Note on the Convergence of a Sequence of Functions of a Certain Type. Read (Chicago) Dec. 30, 1907. *Annals of Mathematics*, ser. 2, No. 3, vol. 9, pp. 123–126 ; Apr., 1908.

CAJORI, F. Notes on the History of the Slide Rule. Read (Southwestern Section) Nov. 30, 1907. *American Mathematical Monthly*, vol. 15, No. 1, pp. 1–5 ; Jan., 1908.

CARMICHAEL, R. D. Multiply Perfect Numbers of Four Different Primes. Read Oct. 27, 1906. *Annals of Mathematics*, ser. 2, vol. 8, No. 4, pp 149–158 ; July, 1907.

—— On Dividing an Angle into Parts Having the Ratios of Any Given Straight Lines. Read Feb. 23, 1907. *American Mathematical Monthly*, vol. 14, Nos. 6–7, pp. 115–117 ; June–July, 1907.

—— On Constructing a Cube Having a Given Ratio to a Given Cube. Read (Chicago) Mar. 30, 1907. *American Mathematical Monthly*, vol. 14, No. 10, pp. 174–176 ; Oct., 1907.

—— Note on Certain Inverse Problems in the Simplex Theory of Numbers. Read Sept. 5, 1907. *Bulletin of the American Mathematical Society*, vol. 14, No. 2, pp. 74–77; Nov., 1907.

—— On Certain Transcendental Functions Defined by a Symbolic Equation. Read Sept. 5, 1907. *American Mathematical Monthly*, vol. 15, No. 4, pp. 78–83; Apr., 1908.

—— On the Classification of Plane Algebraic Curves Possessing Fourfold Symmetry with Respect to a Point. Read Sept. 5, 1907. *Annals of Mathematics*, ser. 2, vol. 9, No. 2, pp. 53–56; Jan., 1908.

—— On a Certain Class of Quartic Curves. Read Oct. 26, 1907. *American Mathematical Monthly*, vol. 15, No. 1, pp. 7–10; Jan., 1908.

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